Answers for Dynamical Systems and Chaos, 2 April 2025.

I.

1. Fixed points: $x^* = \frac{1 \pm \sqrt{1+8r-4r^2}}{2r}$
2. The fixed points exist when the terms under the suqareroot are positive, which determines the bifurcations: $r_{c1} = 1 - \frac{1}{2}\sqrt{5}, r_{c2} = 1 + \frac{1}{2}\sqrt{5}$.
3. Stability of fixed points: $f'(x) = 2rx - 1 \Rightarrow f'(x^*) = \pm\sqrt{1+8r-4r^2}$. (+) is unstable, (-) is stable
4. Behaviour around r $\simeq$ 0: $x^* = \frac{1 \pm \sqrt{1+8r-4r^2}}{2r} \simeq \frac{1 \pm (1+4r)}{2r} = \frac{1}{r} + 2$ or -2. The bifurcation diagram (r, x^*) passes the x^* axes at $x^* = -2$. When $r \to 0^+$, x^* diverges to infinity. When $r \to 0^-$, x^* diverges to minus infinity.
5. At $r_{c1} = 1 - \frac{1}{2}\sqrt{5}$ there is a saddle node bifurcation. At $r_{c2} = 1 + \frac{1}{2}\sqrt{5}$ there is a saddle node bifurcation.
6. At $r_{c1} = 1 - \frac{1}{2}\sqrt{5}$ there is a saddle node bifurcation where the 'top' branch is stable, while the 'down' branch going to minus infinity is unstable. At $r_{c2} = 1 + \frac{1}{2}\sqrt{5}$ there is a saddle node bifurcation where the 'top' branch going to plus infinity is unstable, while the 'down' branch is stable.

II.

7. $(x^*, y^*) = (\frac{1}{\mu}, \mu)$.
8. $\lambda(\mu) = \frac{1-\mu^2 \pm \sqrt{1-6\mu^2+\mu^4}}{2}$. In Strogatz notation: $\tau = 1 - \mu^2, \quad \Delta = \mu^2$.
9. When the expression under the square-root changes sign, the behavior of the fixed point changes, $\mu^2 = 3 + \sqrt{8}$.
10. At $\mu^2 = 3 - \sqrt{8}$ where the fixed point changes from being an unstable spiral to an unstable node.
11. The real part changes sign at $\mu = 1$ which is where the Hopf bifurcation takes place.

III.

12. $x^* = x^* + r - x^{*2} \Rightarrow x^* = \pm\sqrt{r}$.
13. Stability: $f'(x^*) = 1 - 2x^* = 1 \mp 2\sqrt{r}$. $+\sqrt{r}$ is stable for $0 \leq r \leq 1$. $-\sqrt{r}$ is unstable for all r.
14. $f'(x) = 1 - 2x = 0 \Rightarrow x_s = \frac{1}{2}$
15. $f'(x^* = +\sqrt{r}) = 1 - 2\sqrt{r} = -1 \Rightarrow r = 1$
16. $x_s = \frac{1}{2}$ must be part of a super-stable two cycle (otherwise the derivative cannot be zero). This two-cycle must therefore fulfill ($\to$ means 'mapped') $\frac{1}{2} \to \frac{1}{4} + r \to \frac{1}{4} + r + r - (r^2 + \frac{r}{2} + \frac{1}{16}) = \frac{1}{2} \Rightarrow 16r^2 + 5 - 24r = 0$.

    This has solutions $r = \frac{5}{4}$ and $r = \frac{1}{4}$. The first value corresponds to the two cycle $\frac{1}{2} \to \frac{3}{2} \to \frac{1}{2}$