

**Exam for Dynamical Systems and Chaos, 2 April 2025.**

**Duration: 4 hours.**

**All questions are equally weighted.**

**Books, notes and computer are allowed. Answers can be written in Danish and English.**

**Results and solutions are posted on the home page.**

**I.**

Consider the differential equation

$$\dot{x} = f(x, r) = rx^2 - x + r - 2, \quad \text{where } r \text{ is a real parameter} \quad (1)$$

1. Find the fixed points  $x^*$  for (1) as a function of  $r$ .
2. Determine two bifurcation points  $r_{c1}$  and  $r_{c2}$  where  $r_{c1} < 0$  and  $r_{c2} > 0$ .
3. From the derivative of  $f(x, r)$ , determine the stability of the fixed points  $x^*$  for (1) as a function of  $r$ .
4. From a Taylor expansion determine the behaviour of the two fixed points around  $r \simeq 0$  (Note:  $r$  can both be positive and negative).
5. What type of bifurcations takes place at  $r_{c1}$  and  $r_{c2}$ .
6. Sketch the bifurcation diagram, i.e. fixed points  $x^*$  as a function of  $r$  (full line: stable fixed points, dashed line: unstable fixed points).

**II.**

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= \mu - xy^2 \\ \dot{y} &= xy^2 - y \end{aligned} \quad \text{with } \mu > 0 \quad (2)$$

7. Find the fixed point for the system (2).
8. Determine the eigenvalues,  $\lambda(\mu)$ , as a function of  $\mu$  of the linearized system for the fixed point.
9. Find the value of  $\mu^2$  where the fixed point changes from being a stable spiral to a stable node.
10. Describe how the behaviour of the fixed point changes at the  $\mu$ -value  $\mu^2 = 3 - \sqrt{8}$ .
11. The system has a limit cycle for some values of  $\mu$ . Find the value of  $\mu$  where there is a Hopf bifurcation.

**III.**

Consider the discrete mapping

$$x_{n+1} = f(x_n) = x_n + r - x_n^2 \quad r \geq 0 \quad (3)$$

12. Determine the fixed points for (3).
13. Determine the stability of the fixed points.
14. Determine the  $x$ -value,  $x_s$ , of the super-stable fixed point.
15. Find the  $r$ -value where a period doubling takes place.
16. Write a second-order equation in  $r$  which determines the  $r$ -value of the super-stable two-cycle (Hint:  $x_s$  is part of a super-stable two-cycle).