Exam for Dynamical Systems and Chaos, 10 April 2024.
Duration: 4 hours.
All questions are equally weighted except question 5 that counts half.
Books, notes and pocket calculator are allowed. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

Consider the differential equation

$$\dot{x} = a - \sin^2(x), \qquad -\pi \leq x \leq \pi \tag{1}$$

1. Find an expression for the fixed points of (1) in terms of the parameter a. For which values of a do the fixed points exist?
2. Find the values $a = a_{c1}$ and $a = a_{c2}$ where bifurcations occur ($a_{c1} < a_{c2}$). What type of bifurcations take place.
3. Determine the stability of the fixed points. (Note: For $a_{1c} < a < a_{2c}$ there are four fixed points).
4. Draw the bifurcation diagram as a function of a (full line: stable fixed point, dashed line: unstable fixed point)
5. Use the equation for the fixed point and perform a small variation around the fixed point $x^* = \pi/2$ so $x = \pi/2 + \epsilon$ and around the bifurcation point $a_{c2} - \delta$ where $|\epsilon| \ll 1, |\delta| \ll 1$. Expand to second order in ϵ and first order in δ and show that $\epsilon \sim \sqrt{\delta}$.

II.

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= -2xy^2 - x^3 + by \\ \dot{y} &= -x + x^2 y \end{aligned} \tag{2}$$

6. Show that $(x^*, y^*) = (0, 0)$ is a fixed point for (2). From the Jacobian use Strogatz formalism to find τ and Δ in the fixed point.
7. Can you determine the stability of $(x^*, y^*) = (0, 0)$ from this?
8. Use the Lyapunov function $V(x, y) = ax^2+y^2$ and determine a set of values (a, b) where $(x^*, y^*) = (0, 0)$ is stable.
9. A bifurcation takes place at a value $b = b_c$. Determine b_c
10. Now set $b = -4$. Find the eigenvalues and eigenvectors at $(x^*, y^*) = (0, 0)$. What type of fixed point is it?

III.

Consider the discrete mapping

$$x_{n+1} = f(x_n) = rx_n + \frac{2}{x_n}, \quad x_n \neq 0 \tag{3}$$

11. Determine the fixed points for (3).
12. Determine an interval in the parameter r where the fixed points are stable.
13. For $x > 0$, determine the value of the parameter r_s where the fixed point is super-stable. Determine the super stable fixed point x_s.
14. Write an equation in x and r for the two-cycle. Solve the equation for x as a function of r to find the points in the two-cycle (which Strogatz calls p,q) (Hint: Multiply the equation by $rx + \frac{2}{x}$ and set $y = x^2$).
15. For $r = -3$, find the two-cycle points. (Note: The two-cycle points must be different and real).