Answers for Exam for Dynamical Systems and Chaos, 8 April 2026.

I.

Consider the differential equation

$$\dot{x} = r \ \cos x \ - \ \sin 2x, \qquad - \pi < x \leq \pi \tag{1}$$

1. Use $\sin 2x = 2\sin x \cos x$ to write: $\dot{x} = \cos x(r - 2\sin x)$ so $\cos x = 0 \Rightarrow \underline{x^* \equiv x^\pm = \pm\pi/2}$
2. Use $f'(x) = -r\sin x - 2\cos 2x$ so $f'(x^+) = -r + 2 \Rightarrow r_+ = 2$ and $f'(x^-) = r + 2 \Rightarrow r_- = -2$.
3. Two new fixedpoints can exist if $r - 2\sin x = 0$. This has only solutions for $-2 \leq r \leq 2$. Here the fixedpoints are $\sin x^* = \frac{r}{2}$
4. We use $f'(x) = -r\sin x - 2\cos 2x = -r\sin x - 2(1 - 2\sin^2 x)$. Inserting $\sin x^* = \frac{r}{2}$, we get: $f'(x^*) = -\frac{r^2}{2} - 2(1 - 2\frac{r^2}{4}) = -2 + \frac{r^2}{2}$. Since $r \leq 2$ these are stable.
5. The two fixed points emerging are stable, so it is supercritical.
6. $r \simeq 2 \ \Rightarrow \ x^* \simeq \pi/2 + x \ \Rightarrow \ \sin(\pi/2 + x) \simeq 1 - \frac{x^2}{2} \ = \ \frac{r}{2} \ \Rightarrow \ x = \pm\sqrt{2 - r} \ \Rightarrow \ x^* \simeq \pi/2 \pm \sqrt{2 - r}$. $(r \leq 2)$.

II.

Consider the system

$$\begin{aligned} \dot{x} &= \beta + \mu x^{1+\alpha} - y x^\alpha \\ \dot{y} &= -x - y \end{aligned} \tag{2}$$

7. One nullcline is $y = x$ (straight line) and the other is $y = \frac{\beta}{x}$.
8. For $\dot{y}$ we have $y = x$. For $\dot{x}$ we obtain $0 = \beta - yx^\alpha$. Inserting $y = x$ yields: $x^* = \beta^{\frac{1}{1+\alpha}}$
9. We calculate $\partial_x(x^{-\alpha}(\beta - yx^\alpha)) + \partial_y(x^{-\alpha}(x - y) = -\beta\alpha x^{-\alpha-1} - x^{-\alpha}$ Since $\alpha > 0$ and $x \geq 0$, this is always negative. Therefore limit cycles cannot exist.
10. For the second equation we again have $x^* = y^*$. Inserting this gives: $\beta + \mu x^{\frac{3}{2}} - x^{\frac{3}{2}} = 0 \Rightarrow x^* = (\frac{\beta}{1-\mu})^{\frac{2}{3}}$
11. We have: $J_{11} = \frac{1}{2}x^{\frac{1}{2}}(3\mu - 1)$, $J_{12} = -x^{\frac{1}{2}}$, $J_{21} = 1$, $J_{22} = -1$.
12. $\Delta = \frac{3}{2}(1 - \mu)^{\frac{2}{3}}\beta^{\frac{1}{3}} > 0$. Therefore it must be a spiral on one side of the bifurcation and for $\tau = 0$ a Hopf bifurcation. Now $\tau = -1 + \frac{1}{2}x^{\frac{1}{2}}(3\mu - 1) \Rightarrow 1 = \frac{1}{2}(3\mu - 1)(\frac{\beta}{1-\mu})^{\frac{1}{3}}$. Isolating β yields: $\beta = \frac{8(1-\mu)}{(3\mu-1)^3}$.
13. Since $\tau = 0$, we need to calculate Δ, since $\omega = \sqrt{\Delta}$. From the Jacobian $\Delta = \frac{3}{2}(1 - \mu)^{\frac{2}{3}}\beta^{\frac{1}{3}}$. For $\mu = \frac{2}{3}$, we obtain $\beta = \frac{8}{3}$ so $\Delta = \frac{3}{2}(\frac{1}{3})^{\frac{2}{3}}(\frac{8}{3})^{\frac{1}{3}} = \frac{1}{2}8^{\frac{1}{3}} = 1$.

III.

Consider the discrete mapping

$$x_{n+1} = f(x_n) = 2 - \mu x_n^2 \tag{3}$$

14. $x^* = 2 - \mu x^{*2} \Rightarrow x^* = \frac{-1\pm\sqrt{1+8\mu}}{2\mu}$.

    Stability: $f'(x^*) = -2\mu x^* = 1 \mp \sqrt{1 + 8\mu}$. The '+' solution is alw ays unstable. The '-' solution becomes unstable when $f'(x^*) = -1 \Rightarrow +1 - \sqrt{1 + 8\mu} = -1 \Rightarrow \mu_1 = \frac{3}{8}$.
15. For a two-cycle: $x_+ = 1 - \mu x_-^2, x_- = 1 - \mu x_+^2 \Rightarrow x_+ - x_- = \mu(x_+ + x_-)(x_+ - x_-) \Rightarrow x_+ + x_- = 1/\mu$.
16. $x_- = 2 - \mu x_+^2 \Rightarrow x_+ + 2 - \mu x_+^2 = 1/\mu \Rightarrow x_\pm = \frac{1\pm\sqrt{8\mu-3}}{2\mu}$.
17. $f^{2\prime}(x_+) = f'(x_-) \cdot f'(x_+) = (-2\mu x_-) \cdot (-2\mu x_+) = 4 - 8\mu = 0 \Rightarrow \mu_s = 1/2$
18. $f(\frac{12}{7}) = 2 - \frac{7}{8}\frac{12}{7}^2 = -\frac{4}{7}$

    $f(-\frac{4}{7}) = 2 - \frac{7}{8}\frac{4}{7}^2 = \frac{12}{7}$

    Yes, it is a two-cycle. For $\mu = \frac{7}{8}$, $f^{2\prime}(\frac{12}{7}) = -3$ so it is unstable.