Answers for Dynamical Systems and Chaos, 10 April 2024.

I.

Consider the differential equation

1. $\sin(\mathrm{x}^*) = \pm\sqrt{\mathrm{a}}$. They exist for $0 \leq a \leq 1$.

2. From above we find $a_{c1} = 0$ and $a_{c2} = 1$ They are all saddle node bifurcations, in $x^* = 0, x^* = \pi/2, x^* = -\pi/2$.

3. $f'(x) = -2\sin(\mathrm{x})\cos(\mathrm{x}) = -2\sin(\mathrm{x}) \pm \sqrt{1-\sin^2(\mathrm{x})} \Rightarrow \mathrm{f}'(\mathrm{x}^*) = -2(\pm\sqrt{\mathrm{a}} \pm \sqrt{1-\mathrm{a}})$. So, For $\pi/2 < x^* < \pi$ (+,-) unstable, for $0 < x^* < \pi/2$ (+,+) stable, for $-\pi/2 < x^* < 0$ (-,+) unstable, $-\pi < x^* < -\pi/2$ (-,-) stable.

4. From above, full line: stable fixed point, dashed line: unstable fixed point. Three saddle node bifurcations in $x^* = 0, x^* = \pi/2, x^* = -\pi/2$.

5. $x^* = \pi/2$ so $\sin(\mathrm{x}) = \pi/2 + \epsilon \approx 1 - \epsilon^2/2$ and $a_{c2} = 1$ so $\sqrt{1-\delta} \approx 1 - \delta/2 \Rightarrow \epsilon \sim \sqrt{\delta}$.

II.

6. $(x^*, y^*) = (0, 0)$ is a fixed point.

$$J = \left\{ \begin{array}{cc} -2y^2 + 3x^2 & -4xy + b \\ -1 + 2xy & x^2 \end{array} \right\} \quad (1)$$

   From the Jacobian we find $\tau = 0$ and $\Delta = b$ in the fixed point.

7. Is marginal because $\tau = 0$. Cannot determine the stability of $(x^*, y^*) = (0, 0)$ from this.

8. $\dot{V}(x, y) = 2ax(-2xy^2 - x^3 + by) + 2y(-x + x^2 y)$. So $\dot{V}(x, y) < 0$ if $a \cdot b = 1$. One solution $a = 1/2, b = 2$.

9. A bifurcation takes place at $b_c = 0$ from eigenvalues $\lambda = \pm\sqrt{-b}$

10. $b = -4 \Rightarrow \lambda = \pm 2$. Eigenvectors $(2, -1), \lambda = 2$ and $(2, 1), \lambda = -2$. Fixed point is a saddle point.

III.

11. $x^* = rx^* + \frac{2}{x^*} \Rightarrow x^* = \pm\sqrt{\frac{2}{1-r}}$

12. $f'(x_n) = r - \frac{2}{x_n^2} \Rightarrow f'(x^*) = 2r - 1$. Stable if $0 \leq r \leq 1$.

13. $2r - 1 = 0 \Rightarrow r_s = \frac{1}{2}$. $x_s = \sqrt{\frac{2}{1-\frac{1}{2}}} = 2$.

14. $f^2(x_n) = r(rx_n + \frac{2}{x_n}) + \frac{2}{rx_n + \frac{2}{x_n}} = x_n \Rightarrow r(rx_n + \frac{2}{x_n})^2 + 2 = x_n(rx_n + \frac{2}{x_n}) \Rightarrow (rx_n + \frac{2}{x_n})^2 = x_n^2$. With $y = x_n^2 \Rightarrow y = \frac{-4r \pm 4}{2(r^2-1)} = \frac{-2r \pm 4}{(r+1)(r-1)} \Rightarrow x_n = \pm\sqrt{\frac{-2}{r+1}}, x_n = \pm\sqrt{\frac{2}{1-r}}$.

15. For $r = -3$ we get $p = 1, q = -1$ for the two-cycle. The other solution gives for $r = -3, x_n = \pm\sqrt{\frac{2}{1+3}}$ which are fixed points.