Answers to Exam for Dynamical Systems and Chaos, 8 April 2015.

I.

1. Three fixed points: $(x^*, y^*) = (0, 0)$, $(x^*, y^*) = (\sqrt{\frac{1+\sqrt{1-4a^2}}{2a}}, \frac{1+\sqrt{1-4a^2}}{2}$, $(x^*, y^*) = (\sqrt{\frac{1-\sqrt{1-4a^2}}{2a}}, \frac{1-\sqrt{1-4a^2}}{2}$.
2. $[0; a_c] = [0; \frac{1}{2}]$.
3. $$J = \left\{ \begin{array}{cc} -2ax & 1 \\ \frac{4x^3}{(1+x^4)^2} & -1 \end{array} \right\} \quad (1)$$
   In $(0, 0)$, $\lambda = 0, -1$. The fixed point is stable (It is isolated so the marginal eigenvalue will not matter).
4. Null-clines : $y = x^2/4, y = \frac{x^4}{1+x^4}$
5. $(x^*, y^*) = (\sqrt{2+\sqrt{3}}, \frac{2+\sqrt{3}}{4}), \sqrt{2-\sqrt{3}}, \frac{2-\sqrt{3}}{4})$ or $\sqrt{2+\sqrt{3}} = \frac{1}{2}(\sqrt{6}+\sqrt{2})$.
6. We get a stable-node and a saddle-point after the bifurcation. This is a saddle-node bifurcation taking place at $a = a_c$ where two fixed points appear "out of the blue".

II.

7. Clearly $(x^*, y^*) = (0, 0)$ is a fixed point. Setting time derivatives to zero and multiplying $\dot{x}$ by y and $\dot{y}$ by x and subtracting results in $0 = axy + y^2 - xy\sqrt{x^2+y^2} + x^2 - axy + xy\sqrt{x^2+y^2} = x^2 + y^2$ for any fixed point which only has the solution $(x^*, y^*) = (0, 0)$.
8. Jacobian in $(0, 0)$:
   $$J = \left\{ \begin{array}{cc} a & 1 \\ -1 & a \end{array} \right\} \quad (2)$$
   $(\lambda - a)^2 + 1 = 0 \Rightarrow \lambda = a \pm i$
9. $a < 0$: fixed point stable, $a = 0$: fixed point marginal, $a > 0$: fixed point unstable.
10. $\dot{V}(x, y) = x\dot{x} + y\dot{y} = (x^2 + y^2)(a - \sqrt{x^2+y^2}) < 0$ for $a < 0$. From Lyapunov function this means $((x^*, y^*) = (0, 0)$ is asymptotically stable.
11. $x\dot{x} + y\dot{y} = r\dot{r} = (x^2 + y^2)(a - \sqrt{x^2+y^2}) = r^2(a - r) \Rightarrow \frac{dr}{dt} = (a - r)r$
12. For $r = \frac{a}{2}, \dot{r} = \frac{a^2}{4} > 0$ so trajectory spiral outwards. For $r = 2a, \dot{r} = -2a^2 < 0$ so trajectory spiral inwards and we have a trapping regime without a fixed point. From the Poincaré Bendixon theorem we know a stable limit cycle exists.

III.

13. $x_n^* = ax_n^* - {x_n^*}^3 \Rightarrow x_n^* = 0, x_n^* = \pm\sqrt{a-1}$. $x_n^* = 0$ exists always, $x_n^* = \pm\sqrt{a-1}$ exist for $a \geq 1$
14. $f'(x_n) = a - 3x_n^2$. $f'(0) = a$, stable for $-1 < a < 1$. $f'(\pm\sqrt{a-1}) = -2a + 3$ is stable for $|-2a+3| < 1 \Rightarrow 1 < a < 2$
15. $f(\sqrt{a+1}) = a\sqrt{a+1} - \sqrt{a+1}^3 = a\sqrt{a+1} - (1+a)\sqrt{a+1} = -\sqrt{a+1}$
16. We need to check whether $|f'(\sqrt{a+1}) \cdot |f'(-\sqrt{a+1})| = |-2a-3|^2 < 1$. Thus the period-2 cycle would be stable for $-2 < a < -1$, however it only exist for $a \geq -1$ and is therefore always unstable.