Answers for Dynamical Systems and Chaos, 11 April 2012.
Duration: 4 hours. Books, notes and computers are allowed.

I.

1. $\dot{x} = -x^2 + x(1-r) + r = 0 \quad \Rightarrow \quad x^* = -r, 1.$ $x^* = -r$ is unstable, $x^* = 1$ is stable.
2. $\dot{x} = -\frac{dv}{dx} \quad \Rightarrow \quad v(x) = \frac{1}{3}x^3 - \frac{1}{2}(1-r)x^2 - rx$
3. $\frac{\dot{x}}{dx} = 0 \quad \Rightarrow \quad x_m = \frac{1-r}{2}$. Inserted into $\dot{x}$: $\dot{x}_m = \frac{1}{4}(1-r)^2 + r \;=\; \frac{(r+1)^2}{4}$.

II.

4. $(0,0)$ inserted gives $(\dot{x}, \dot{y}) = (0,0)$.
$$J = \left\{ \begin{array}{cc} -3x^2 & 1 \\ -a & -b \end{array} \right\} \tag{1}$$
$\lambda = \frac{1}{2}(-b \pm \sqrt{b^2 - 4a})$.
5. $a > 0, b > 0, \quad \tau = -b, \quad \Delta = a$ Always linearly stable because $\tau < 0$. Spiral occurs when $b^2 - 4a < 0$
6. $V(0,0) = 0, V(x,y) > 0$ for $(x,y) \neq (0,0)$.
$\dot{V}(x,y) = 2x\dot{x} + 2y\dot{y} = -2x^4 + 2xy - 2axy - 2by^2$.
For $a = 1$, $\dot{V}(x,y) = -2x^4 - 2by^2 < 0$.
7. $a = 1 \quad \Rightarrow \quad \tau = -b, \Delta = 1$. Complex eigenvalues will go from a stable spiral to an unstable spiral, when b does from positive to negative values, i.e. a Hopf bifurcation. It is super-critical as from Lyapunov function we found $(0,0)$ is globally stable for $b > 0$ and therefore there cannot be an unstable limit cycle around the fixed point $(0,0)$.
8. $x^* = \sqrt{-a}, y^* = -a\sqrt{-a} = \sqrt{-a^3}$ and $x^* = -\sqrt{-a}, y^* = a\sqrt{-a} = -\sqrt{-a^3}$ $(a < 0)$.
9. $\tau = 3a - 1, \Delta = -2a$. When $a < 0$ the two new fixed points are stable. At the same time the fixed point in $(0,0)$ becomes unstable: a super-critical pitch-fork bifurcation.

III.

10. $x^* = (x^*)^2 + c \quad \Rightarrow \quad x^* = \frac{1}{2}(1 \pm \sqrt{1-4c})$
11. $c < \frac{1}{4} : x'_{n+1} = 2x^* = 1 \pm \sqrt{1-4c}$: '+'-solution is always unstable, '-'-solution is stable for $-\frac{3}{4} < c < \frac{1}{4}$ and unstable for $c < -\frac{3}{4}$.
12. For $c = -\frac{3}{4}$ then $x'_{n+1} = 1 - \sqrt{1-4c} = -1$. A period-doubling takes place.
13. $(x_n^*, y_n^*) = (\frac{1}{2}, \pm\sqrt{c - \frac{1}{4}})$.
14. $c \in [\frac{1}{4}; \infty]$.
15.
$$J = \left\{ \begin{array}{cc} 2x_n & -2y_n \\ 2y_n & 2x_n \end{array} \right\} \tag{2}$$
$\lambda = 1 \pm \sqrt{1-4c}$
16. $z_{n+1} = x_{n+1} + iy_{n+1} = z_n^2 + c = (x_n + iy_n)(x_n + iy_n) + c = x_n^2 - y_n^2 + 2ix_ny_n + c \quad \Rightarrow$
$x_{n+1} = x_n^2 - y_n^2 + c, \quad y_{n+1} = 2x_ny_n$