Exam for Dynamical Systems and Chaos, 12 April 2023.
Duration: 4 hours. All questions are equally weighted except 8. and 9. that counts half.
Books and notes are allowed. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

Consider the differential equation

$$\dot{x} = f(x, r) = rx \; - \; 2x^2 \; + \; x^3, \qquad \text{where } r \text{ is a real parameter} \tag{1}$$

1. Find the fixed points for (1) (some are a function of r).
2. Determine a bifurcation point $r_{c1} > 0$ and find the value of the fixed point x^* at the bifurcation point.
3. From the derivative of $f(x, r)$, determine the stability of the fixed points for (1) as a function of r.
4. What type of bifurcation takes place at r_{c1}.
5. Another bifurcation takes place at $r_{c2} = 0$. What type of bifurcation is that.
6. Sketch the bifurcation diagram, i.e. fixed points x^* as a function of r for $r \leq r_{c1}$ (full line: stable fixed points, dashed line: unstable fixed points).

II.

Consider the system in polar coordinates (r, θ)

$$\dot{r} = r(\mu - r^2), \qquad \dot{\theta} = 1 + \gamma r \sin(\theta) \tag{2}$$

where μ, γ are two real parameters.

7. Find the fixed points of the system and determine their range of existence in terms of μ, γ.
8. Use the Poincaré-Bendixson theorem, and the equation $\dot{r} = r(\mu - r^2)$ to show that a stable limit cycle exists when $0 < \mu < 4$ and $\gamma = 1/2$. (Hint: Find a radius r_{max} where the trajectory spirals inwards and a radius r_{min} where the trajectory spirals outwards. Note that there is not a unique choice of r_{max} and r_{min}).
9. Determine the stability of the fixed point in $r^* = 0$ or $(x^*, y^*) = (0, 0)$. (Hint: from $(x, y) = (r\cos(\theta), r\sin(\theta))$ calculate $(\dot{x}, \dot{y})$ by inserting Eqs. (2) and keep only linear terms to find the Jacobian).
10. What kind of bifurcation occurs when μ passes from negative to positive?

III.

Consider the discrete mapping

$$x_{n+1} = f_r(x_n) = rx_n + x_n^3 \tag{3}$$

11. Determine the fixed points x_n^* of (3).
12. From the derivative of (3) show that $x_n^* = 0$ is a stable fixed point in an interval $[r_-, r_+]$ of the parameter.
13. Determine the stability of the two non-trivial fixed point (when they exist).
14. Describe the bifurcation that takes place for $x_n^* = 0$ around $r \simeq r_+$.
15. Describe the bifurcation that takes place for $x_n^* = 0$ around $r \simeq r_-$.
16. Sketch the bifurcation diagram (r, x_n^*) (full line: stable fixed points, dotted line: unstable fixed points).