**Exam for Dynamical Systems and Chaos, 6 April 2022.**
**Duration: 4 hours. All questions are equally weighted.**
**Books and notes are allowed. Answers can be written in Danish and English.**
**Results and solutions are posted on the home page.**

## I.

Consider the differential equation

$$\dot{x} = f(x,r) = \cos x \;-\; r\,\sin 2x, \qquad -\pi \le x < \pi \tag{1}$$

1. Determine the two fixed points $x_1^* > 0, x_2^* < 0$ of (1) that exist for any value of the parameter $r$ (Hint: Use standard trigonometric relations).
2. Determine the stability of $x_1^*$ and $x_2^*$ as a function of $r$.
3. Show that both fixed points undergo a bifurcation at the values $r = r_{c1}$ (for $x_1^*$) and $r = r_{c2}$ (for $x_2^*$).
4. Show that the bifurcations give rise to 'new' fixed points that satisfy $\sin x^* = 1/2r$. Determine the intervals where these fixed points exist.
5. Determine the stability of these fixed points. What type of bifurcations take place at $r_{c1}$ and $r_{c2}$?
6. Sketch the bifurcation diagram $x^*$ versus $r$ for $x^* \in [-\pi;\pi]$ (full line: stable fixed points, dashed line: unstable fixed points).

## II.

Consider the Fitz-Hugh Nagumo model

$$\dot{u} = u - \frac{1}{3}u^3 - w + I, \qquad \dot{w} = \frac{1}{2}(1 + u - w) \tag{2}$$

with $I \in R$ being the constant current injected in the neuron.

7. Sketch the nullclines of the system for I = 0.
8. Find the fixed point $(u^*, w^*)$ of the system in terms of I.
9. By linearization, in particular using the $\tau, \Delta$ formalism of Strogatz, determine for which values of I the fixed point is stable and for which values of $I$ it is unstable. Then classify the fixed point for the particular case I = 4/3.
10. Show that linearization fails when I = 1.
11. Assuming that we know $(u^*, w^*)$ is a repeller (i.e. an unstable fixed point) when I = 1, show that a limit cycle exists around it by using the Poincaré-Bendixson theorem. (Hint: draw the nullclines in $u, w$ space and construct a square centered in $(0,0)$ with sides at $u = \pm 3$ and $w = \pm 4$).
12. Given the information you have collected on the previous points, what type of bifurcation could the system undergo?

## III.

Consider the discrete one-dimensional map

$$x_{n+1} = f(x_n) = \begin{cases} 3x_n, & \text{if } \; 0 \le x_n \le \frac{1}{3} \\ \frac{3}{2}(1 - x_n), & \text{if } \; \frac{1}{3} \le x_n \le 1 \end{cases} \tag{3}$$

13. Find the fixed points of (3) and determine their stability .
14. Draw the cobweb and the time-series of the map (3). (Hint: Start from a point relatively close to the fixed point and draw 7-10 steps for the coweb and 4-5 steps for the time-series).
15. Which features of the figures give you information about the location and the stability of the fixed points? Do the features and information depend on the stability of the fixed point?
16. Determine the points of the two-cycle $x_1^* \to x_2^* \to x_1^* ....,\ (x_1^* \ne x_2^*)$. Is the two-cycle stable or unstable?
17. Show that the Lyapunov exponent of the map is equal to $\lambda = \ln(3) - \frac{2}{3}\ln(2)$. (Hint: The map (3) distributes the points $x_i$ uniformly in $[0;1]$. That means the fraction of points $x_0 \to x_1 \to x_3 \to ....\ x_n$ within each of the two intervals in (3) will be proportional to the length of the interval.)