Exam for Dynamical Systems and Chaos, 14 April 2021.
Duration: 4 hours for problem solving, 1 hour for collecting files and uploading (5 hours total).
All questions are equally weighted.
Books and notes are allowed. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

Consider the differential equation which represents a neuronal model

$$\dot{\theta} = (1 - \cos\theta) + (1 + \cos\theta)I, \tag{1}$$

with $I \in R$ being the constant current injected in the neuron.

1. Show that this system can be regarded as a vector field on a circle.
2. Find a relation for the fixed points in terms of I (maybe use $\frac{1-\cos(x)}{1+\cos(x)} = \tan^2(x/2)$).
3. Show that the fixed points undergo a bifurcation. What type of bifurcation is it?
4. Sketch the fixed points on the circle as I goes through the bifurcation. Indicate the stability of the fixed points by linear stability analysis.
5. Show that by the change of variable $u(t) = \tan(\frac{\theta}{2})$ and using trigonometric relations one can retrieve the normal form $\dot{u}(t) = u^2 + I$ of a known bifurcation.
6. Show that the solution to $\dot{u}(t)$ (with initial value $u(0) = 0$) experiences a blow-up in finite time (T_{blow}) and find T_{blow}.
7. Now consider the case in which the current I is not constant but grows linearly in time ($I(t) = at$) with a $=$ const. Show that the system has no fixed points for any value of a.

II.

A two-dimensional differential equation system is defined by:

$$\dot{x} = -2x - ay \quad , \quad \dot{y} = -ax - 2by \quad \text{where } a,\ b \text{ are real parameters and } b > 0\ , \tag{2}$$

8. Show by integration that (2) is a gradient system and determine the potential $V(x, y)$ (ignore integration constants).
9. What does that say about the solutions to (2).
10. Determine the Jacobian of (2) and derive the eigenvalues $\lambda_\pm$ of the fixed point as a function of a and b.
11. For $b = 1$ describe the nature of the fixed point for all values of a.
12. For $a = \sqrt{7}$ and $b = 4$ find the eigenvalues and eigenvectors and sketch the flow.
13. For $a = 4$ and $b = 1$ find the eigenvalues and eigenvectors and sketch the flow.

III.

Consider a 1-d map

$$x_{n+1} = f(x_n) = x_n + \frac{a}{2}\frac{\cos(x_n)}{\sin^2(x_n)} \tag{3}$$

where a is a real parameter and $x_n \in]0; \pi[\cup]\pi; 2\pi[$.

14. Find the fixed points x_1^*, x_2^* of (3).
15. Find the derivative $f'(x_n)$ of the map (3).
16. With $x_1^* < x_2^*$ determine the intervals $[a_{1,min}, a_{1,max}]$ and $[a_{2,min}, a_{2,max}]$ for which the fixed points are stable.
17. For fixed point x_1^* determine the value $a = a_{ss}$ where it is super stable and the value $a = a_{PD}$ where a period doubling takes place.
18. At the period doubling point $a = a_{PD}$, make a little perturbation $x_0 = x_1^* + \epsilon$ and show that to linear order in ϵ a two cycle $x_0 \to x_1 \to x_2 = x_0$ exists (Hint: Expand $\sin(x_n)$ and $\cos(x_n)$ around x_1^* to linear order in ϵ and keep all terms to linear order).