Exam for Dynamical Systems and Chaos, 1 April 2020.
Duration: 4 hours for problem solving, 1 hour for collecting files and uploading (5 hours total).
All questions are equally weighted except question 16 that counts half.
Books and notes are allowed. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

Consider the differential equation

$$\dot{x} = f(x, r) = \frac{\cos x}{\sin x} - r \sin 2x, \qquad 0 \le x < 2\pi \tag{1}$$

1. Show that $x^* = \frac{\pi}{2}$ and $x^* = \frac{3\pi}{2}$ are fixed points of (1) for any value of the parameter r (Hint: Use standard trigonometric relations).
2. Show that both fixed points undergo a bifurcation at the same parameter value $r = r_c$ and determine r_c.
3. Show that the bifurcations give rise to fixed points that satisfy $\sin^2 x^* = \frac{1}{2r}$.
4. Determine the stability of these fixed points using this relation.
5. What type of bifurcations takes place at r_c. Are they super- or sub-critical ?
6. Sketch the bifurcation diagram for $r > 0$ and $x^* \in [0; 2\pi]$ (full line: stable fixed points, dashed line: unstable fixed points). What happens as $r \to \infty$?

II.

A two-dimensional differential equation system is defined by:

$$\dot{x} = x(1 - (ax + y^2)) \quad , \quad \dot{y} = y(1 - (x + y)) \tag{2}$$

where $a > 0$ is a real positive parameter.

7. With the function $g(x, y) = \frac{1}{xy}$, use Dulac criterion to show that there do not exist a limit cycle in the first quadrant, $(x > 0, y > 0)$.
8. Now we consider $(x \ge 0, y \ge 0)$. Find the fixed points of (2) (some depend on the parameter a).
9. Derive the Jacobian matrix of (2).
10. In the following we consider $a = \frac{3}{2}$. Find the eigenvalues of the fixed points and classify the fixed points.
11. Sketch the nullclines and the flow around the fixed points in $(x \ge 0, y \ge 0)$.

III.

Consider a 1-d map

$$x_{n+1} = f(x_n) = x_n + \log(r(2 - x_n)) \tag{3}$$

where r is a real parameter.

12. Find the fixed point x_n^* of (3) as a function of r. Find the derivative $f'(x_n)$ of the map (3).
13. Determine the interval $[r_{min}, r_{max}]$ for which the fixed point x_n^* is stable.
14. Determine the value $r = r_{ss}$ where the fixed point is super stable.
15. Determine the value $r = r_{PD}$ and of the fixed point $x_n = x_{PD}^*$ where a period doubling takes place.
16. At the period doubling point, make a little perturbation $x_0 = x_{PD}^* + \epsilon$ and show that to linear order in ϵ a two cycle $x_0 \to x_1 \to x_2 = x_0$ exists. (Hint: Expand the log function around 1, i.e $\log(1 \pm \delta) \approx \pm\delta$)).