Exam for Dynamical Systems and Chaos, 5 April 2017.
Duration: 4 hours. Books, notes and computers are allowed.
All questions are equally weighted. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

Consider the following set of differential equations

$$\begin{aligned} \dot{x} &= -x + y + xy \\ \dot{y} &= x - y - x^2 - y^3 \end{aligned} \tag{1}$$

1. The system (1) has one fixed point (x^*, y^*). Find it.
2. Derive the Jacobian matrix.
3. Determine the eigenvalues and corresponding eigenvectors for (x^*, y^*).
4. What can you say about the stability of (x^*, y^*).
5. Now argue the fixed point (x^*, y^*) is stable by applying a Lyapunov functional of the form $V(x, y) = ax^2 + 2y^2$ and determine a value of a that ensures the stability (Hint: Collect terms to a quadratic form).
6. Is a limit cycle possible in the system ?

II.

Consider the predator-prey system

$$\begin{aligned} \dot{x} &= x(x(1-x) - y) \\ \dot{y} &= y(x - a), \qquad 0 < a < 1 \end{aligned} \tag{2}$$

7. Find and draw the null-clines of (2).
8. Find the fixed points.
9. Argue from the fixed points alone that a state exists where the predator goes extinct and the prey is still alive.
10. Discuss the stability of the fixed points where one or two of the species are extinct.
11. A Hopf bifurcation for the fixed point $x^* > 0, y^* > 0$ occurs at a critical values $a = a_c$. Determine a_c (Hint: Use Strogatz τ, Δ notation).
12. Show that the fixed point $x^* > 0, y^* > 0$ is unstable for $0 < a < a_c$

III.

Consider a population dynamics model that is written as the 1-d map

$$x_{n+1} = f(x_n) = x_n \exp(r(1 - x_n)) \tag{3}$$

where r is a real parameter.

13. Show that the map always has an extremal point $x_m = \bar{x}$. Determine for which values of r it is a maximum and for which values of r it is a minimum.
14. Find the fixed points x^* of (3).
15. Determine the the stability interval of the fixed points as a function of r.
16. Determine the parameter value $r = r_p$ where one of the fixed points undergoes period doubling.
17. Write the superstable two-cycle as a function of r and show that it leads to the condition
$2r - 1 - \exp(r - 1) = 0$.
18. Find a trivial solution $r = r_s$ to this equation. Which two-cycle does that correspond to ?