**Exam for Dynamical Systems and Chaos, 13 April 2016.**
**Duration: 4 hours. Books, notes and computers are allowed.**
**All questions are equally weighted. Answers can be written in Danish and English.**
**Results and solutions are posted on the home page.**

## I.

Consider the two-dimensional differential equation system

$$\begin{aligned} \dot{x} &= ax + y + x^3 \\ \dot{y} &= x - y \end{aligned} \tag{1}$$

1. Sketch the null clines of the system (1) for $a = -2$.
2. Find the fixed points for (1) as a function of $a$
3. Determine a bifurcation point $a = a^*$
4. Find the Jacobian matrix.
5. Determine the eigenvalues of the fixed points as a function of $a$.
6. Evaluate the stability of the fixed points in $a = -2$ and the stability for the trivial fixed point in $a = 2$.
7. What kind of bifurcation takes place at $a = a^*$.

## II.

For the second-order differential equation

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \tag{2}$$

where $h(x, \dot{x}) = (x^2 - a)\dot{x} + (\dot{x}^2 - b)x \quad , \quad a > 0 \quad , b > 0$

8. Use averaging theory (or equivalently two timing) of (2) to derive an equation for $r'$.
9. Find the fixed points for $r(T)$ in terms of $a$.
10. For $a = 2$ and with the initial condition $r(0) = 1$, show after integrating by partial fraction the solution is $r(T) = \sqrt{\frac{8}{1+7e^{-2T}}}$. Sketch the graph of $r(T)$ and determine the value of $r(T)$ as $T \to \infty$.
11. Use averaging theory (or equivalently two timing) to derive an equation for $\phi'$.
12. Insert the stable fixed point for $r$ and determine an equation for $\phi'$ in terms of $a$ and $b$.
13. What is the resulting frequency $\omega$ for the limit cycle.

## III.

Consider the 1-d map

$$x_{n+1} = f(x_n) = x_n + a\ \cos(x_n)\sin(x_n) \tag{3}$$

14. Find the fixed points $x^*$ of (3) in the interval $[0; \pi[$.
15. Determine the derivative of the map.
16. For the fixed point $x_2^*$ (where $x_1^* < x_2^*$) determine the value of $a$, $a_2$, where the fixed point undergoes period doubling.
17. For the other fixed point $x_1^*$ a period doubling occurs at $a_1 = -2$.

    For $a = a_1$, expand the map (3) to linear order (i.e. $f(x_n) \approx f(x_1^*) + f'(x_1^*) \cdot (x_n - x_1^*)$) in $x_n$. Consider and small variation around $x_1^*$, i.e. $x_1^* = \epsilon$ where $|\epsilon| << 1$, and show that a two cycle for the map exists at this order.