Exam for Dynamical Systems and Chaos, 8 April 2015.
Duration: 4 hours. Books, notes, pencils and computers are allowed.
All questions are equally weighted. Answers can be written in Danish and English.
Results and solutions are posted on the home page.

I.

A protein x can promote itself by binding four molecules at its promotor at the DNA and produce the associated mRNA y leading to the coupled system of differential equations:

$$\dot{x} = y - ax^2 \quad , \quad \dot{y} = \frac{x^4}{1+x^4} - y \quad , \quad a \geq 0, \quad x \geq 0, \quad y \geq 0 \tag{1}$$

1. Find the the fixed points of (1).
2. Determine the interval $[0; a_c]$ where the non-trivial fixed points exist.
3. Determine the Jacobian and find the eigenvalues for the trivial fixed point $(0, 0)$.
4. Sketch the null-clines for $a = \frac{1}{4}$ in the area $x \geq 0, y \geq 0$ and add arrows for the flow.
5. Derive the fixed points for $a = \frac{1}{4}$ for $x > 0, y > 0$.
6. Argue that a saddle-node bifurcation takes place at $a = a_c$.

II.

Consider the coupled system of differential equations:

$$\dot{x} = ax + y - x\sqrt{x^2 + y^2} \quad , \quad \dot{y} = -x + ay - y\sqrt{x^2 + y^2} \tag{2}$$

7. Multiply $\dot{x}$ by y and $\dot{y}$ by x and subtract to show that $(x^*, y^*) = (0, 0)$ is the only fixed point.
8. Determine the Jacobian in $(0, 0)$ and determine the eigenvalues of the fixed point (note: keep only terms that do not involve x and y as they will be zero when $(x^*, y^*) = (0, 0)$).
9. What can you conclude about the stability of the fixed point for $a > 0, a = 0, a < 0$ from these eigenvalues.
10. Now use the Lyapunov function $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Show that when $a < 0$ then $\dot{V}(x, y) < 0$ when $(x, y) \neq (0, 0)$. Use this result to argue that that the fixed point is asymptotically stable.
11. Use polar coordinates $x = r\cos\theta, y = r\sin\theta$ and show that one obtains the following equation for r

$$\frac{dr}{dt} = (a - r)r \quad , \quad (\text{Hint : use } \; x\dot{x} + y\dot{y} = r\dot{r}) \tag{3}$$

12. One can show $\dot{\theta} = -1$. Now consider $a > 0$. Insert $r = \frac{a}{2}$ and $r = 2a$ and use Eq. (3) to argue for a trapping region. Apply the Poincaré-Bendixon theorem to argue that there exists a stable limit cycle for all $a > 0$.

III.

Consider the 1-d map

$$x_{n+1} = f(x_n) = ax_n - x_n^3 \tag{4}$$

13. Find the fixed points x_n^* for Eq. (4) and determine for what values of a each fixed point exists.
14. Determine the intervals in a where the fixed points are stable.
15. Verify that a period-2 cycle of $f(x_n)$ is given by $\{\sqrt{1+a}, -\sqrt{1+a}\}$.
16. Determine the stability of this period-2 cycle.