Exam for Dynamical Systems and Chaos, 2 April 2014.
Duration: 4 hours. Books, notes, pencils and computers are allowed.
All questions are equally weighted. Answers can be written in Danish and English.

I.

Consider the differential equation

$$\dot{x} \; = \; 4x^3 + r^2 x - rx \tag{1}$$

1. Find the the fixed points of (1).
2. Classify the fixed points according to their stability.
3. Determine the two bifurcation points r_1 and r_2 (where $r_1 < r_2$. What type of bifurcations occur at these points.
4. Sketch the bifurcation diagram (fixed points versus r) and add the arrows of the flow (Full lines: stable fixed points, dashed lines: unstable fixed points).

II.

Consider the coupled system of differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \epsilon y^3(1 + x^2) \quad , \quad \epsilon > 0 \end{aligned} \tag{2}$$

5. Find the fixed point of the system (2). Determine the eigenvalues of the fixed point.
6. What can you conclude about the stability of the fixed point from these eigenvalues.
7. Now use the Lyapunov function $V(x, y) = \frac{1}{2}x^2 + \frac{1}{2}ay^2$. Derive the equation for $\dot{V}(x, y)$ as a function of a.
8. Determine a value of a such that $\dot{V}(x, y) < 0$ when (x, y) is away from the fixed point. Use this result to argue that that the fixed point is asymptotically stable.
9. Write (2) on the form $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$ and use averaging theory to derive an equation for $r'(T)$.
10. Show that $r(T) \to 0$ as $T \to \infty$. What does that tell us about the fixed point.

III.

Consider the 1-d map

$$x_{n+1} = f(x_n) = rx_n(1 - x_n^2) \tag{3}$$

11. Find the fixed points x_n^* for (3).
12. Determine the stability as a function of r for the trivial fixed point $x_n^* = 0$.
13. Determine the stability as a function of r for the non-trivial fixed points and the intervals for which they are defined.
14. What happen to the non-trivial fixed points when $r \to 0^-$.
15. Three bifurcations take place for the fixed points, at $r_1 < 0$ and at $0 < r_2 < r_3$. Determine the value of r_1, r_2, r_3.
16. What kind of bifurcations take place at r_1 and r_3, respectively ?
17. Draw the bifurcation diagram (r, x_n^*) (full line: stable fixed points, dotted line: unstable fixed points).