Exam for Dynamical Systems and Chaos, 10 April 2013.
Duration: 4 hours. Books, notes, pencils and computers are allowed.
All questions are equally weighted. Answers can be written in Danish and English.

I.

Consider the differential equation

$$\dot{x} = \frac{1}{2r}x^2 - \frac{1}{r}x - 1 \;\; , \;\; r \neq 0 \tag{1}$$

1. Find the the fixed points of (1).
2. The fixed points only exist above a bifurcation point $r_c < 0$. Determine r_c. What type of bifurcation occurs at this point.
3. Classify the fixed points according to their stability for $r < 0$ and $r > 0$, respectively.
4. Sketch the bifurcation diagram (fixed points versus r) and add the arrows of the flow (Full lines: stable fixed points, dashed lines: unstable fixed points). Describe in words what happens as r passes through $r = 0$.

II.

The Lotka-Volterra Predator Prey model describes a simple ecological model between two species:

$$\begin{aligned} \dot{x} &= \alpha x - xy \\ \dot{y} &= xy - \beta y \end{aligned} \tag{2}$$

Where $\alpha, \beta > 0$ are interaction parameters and $x, y \geq 0$ are variables describing the amount of predators and preys on an island.

5. Determine which of the two equations describes the prey and which describes the predator and briefly argue for your answer.
6. Is chaotic motion possible for the system (2) ?
7. Find the fixed points of (2).
8. Determine the linear stability of the fixed point at the origin.
9. Determine the linear stability of the second fixed point and find the roots $\lambda_{1,2}$. What type is the second fixed point according to linear stability analysis ?
10. All trajectories in the phase space for $x, y \geq 0$ can be described by a constant of motion $E(x, y) = y^\alpha e^{-y} x^\beta e^{-x}$. Show that E is conserved (Hint: $\dot{E} = \frac{dE}{dx}\dot{x} + \frac{dE}{dy}\dot{y}$).
11. Draw the nullclines and the fixed points in the (x, y) phase plane and determine the rotation direction around the second fixed point.

    It turns out that the second fixed point is a local minimum of $-E$. Given our knowledge of conservative systems what can now be concluded about the second fixed point ?

III.

Consider the 1-d map

$$x_{n+1} = f(x_n) = \begin{cases} 10x_n + 10 & \text{if} \quad 0 \leq x_n < 2 \\ \frac{1}{2}x_n & \text{if} \quad 2 \leq x_n \end{cases} \tag{3}$$

12. Show there are no fixed point to (3) ? (If you plot the map, dont worry that it is discontinuous).
13. Given any $x_0 \in [1; 2[$, determine the number of iterations m where x_m for the first time comes back in the interval $[1; 2[$.
14. Find the points of the 5-cycle for which $x_0 = x_5$ and $x_0 \in [0; 2[$. Is the 5-cycle stable ?