Exam for Dynamical Systems and Chaos, 11 April 2012.
Duration: 4 hours. Books, notes and computers are allowed.
Exam consists of 3 exercises. The questions are equally weighted.
Answers can be written in Danish and English. Pencil is allowed
Results and solutions are posted on the home page.

I.

Consider the differential equation

$$\dot{x} = -x^2 + x(1-r) + r \quad , \quad r > 0 \tag{1}$$

1. Find the fixed points of (1) and their stability.
2. Determine the potential function $V(x, r)$ and plot it for $r = 2$.
3. Find the maximal positive velocity as a function of r.

II.

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= -x^3 + y \\ \dot{y} &= -ax - by \end{aligned} \tag{2}$$

4. Show that $(0, 0)$ is a fixed point. Find the Jacobian and calculate the eigenvalues of the fixed point $(0, 0)$ as a function of a, b.
5. Show that the fixed point $(0, 0)$ is linearly stable for all values $a > 0, b > 0$.
6. For $b > 0$, use the Lyapunov function $V(x, y) = x^2 + y^2$ to find at least one value of the parameter a where the fixed point $(0, 0)$ is globally stable.
7. Now set $a = 1$. Argue that a Hopf bifurcation takes place for $b = 0$ (hint: you may use the τ, Δ classification scheme). Is the bifurcation super- or sub-critical ?
8. Now set $b = 1$. When $a < 0$ there exist two fixed points different from $(0, 0)$. Find these fixed points.
9. Insert these fixed points into the Jacobian and find the trace and the determinant. What kind of bifurcation takes place at $a = 0$?

III.

Consider the two-dimensional map

$$x_{n+1} = x_n^2 - y_n^2 + c \quad , \quad y_{n+1} = 2x_n y_n \tag{3}$$

10. First consider the case $y_n = 0$ where (3) reduce to the 1d map $x_{n+1} = x_n^2 + c$. Find the fixed points x^*.
11. Determine the stability of the fixed points in the interval of c for which they exist.
12. What happens to the fixed point $x^* < \frac{1}{2}$ for $c = -\frac{3}{4}$.
13. Now consider the full two-dimensional system (3). Find the fixed points (x_n^*, y_n^*) for which y_n is non-zero.
14. For which interval in c do these fixed point exist ?
15. Determine the Jacobian matrix and find the eigenvalues of the fixed points found in question 13.
16. Show that the map (3) can be written as $z_{n+1} = z_n^2 + c$ in the complex variable $z_n = x_n + iy_n$.