Exam for Dynamical Systems and Chaos, 6 April 2011.
Duration: 4 hours. Books, notes and computers are allowed.
Exam consists of 3 exercises. The questions are equally weighted.
Answers can be written in Danish and English. Pencil is allowed
Results and solutions are posted on the home page.

I.

Consider the differential equation

$$\dot{x} = f(x) = r + x - \frac{1}{2}\ln(1 + 2x) \qquad (1)$$

1. Show that $f(x)$ has a minimum for all values of r and determine the point $\tilde{x}$ where $f(x)$ has a minimum.
2. A bifurcation takes place at $r = r_c$. Determine r_c.
3. Around the bifurcation point Taylor expand $\ln(1 + 2x)$ to order x^2 and find the fixed points.
4. What type of bifurcation takes place.
5. Determine the stability of the fixed points found in 3.
6. Draw the bifurcation diagram (full line: stable fixed points, dashed line: unstable fixed points)

II.

For the second-order differential equation

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \qquad (2)$$

where $h(x, \dot{x}) = (x^2\dot{x}^2 - ax^2)\dot{x}$, $a > 0$

7. Use averaging theory (or equivalently two timing) to derive an equation for r'.
8. Find the fixed points for $r(T)$.
9. Determine the stability of the fixed point $r^* > 0$ for all values of $a > 0$?

III.

Consider the discrete tent map

$$x_{n+1} = f(x_n) = \begin{cases} \frac{5}{2}x_n & \text{if} \quad 0 \leq x_n \leq 1/2 \\ \frac{5}{2}(1 - x_n) & \text{if} \quad 1/2 \leq x_n \leq 1 \end{cases} \qquad (3)$$

10. Determine the fixed points of (3) and their stability.
11. Determine the Lyapunov exponent of the map (3).

    Notice, that some point will get mapped outside the interval $[0; 1]$ (i.e. $f(x_n) > 1$ for some x_n). We say these points escape.
12. Find the set of initial conditions x_0 that escape after one iteration.
13. Find the set of initial conditions x_0 that escape after two iterations. (hint: you should find an interval to the left of $x_n = 0.5$ and an interval to the right of $x_n = 0.5$).

    If $f^n(x_0) > 1$ and $0 \leq f^k(x_0) \leq 1$ for all $k < n$ we say x_0 has escaped after n iterations. Some points will never escape. These form a fractal set.
14. Determine similarity dimension of this set using the information in 12 or 13.