Eksamensopgaver til Fysik 222, 10. juni 2004.
Varighed 3 timer. Hjælpemidler tilladt. Eksamen består af 3 opgaver.
Spørgsmål 1-14 indgår med lige vægt i bedømmelsen.
Eksamensresultat og løsninger opslås på hjemmesiden.

I.

Under the spreading of an epidemic disease, let $H(\tau)$ be the number of healthy people and $S(\tau)$ be the number of sick people. The healthy people reproduce at a rate $a > 0$, the death rate for the sick people is $b > 0$, and the incubation rate for the healthy people is proportional with the number of sick people with a rate $c > 0$. This lead to the following system of differential equations

$$\begin{aligned} \frac{dH}{d\tau} &= aH - cHS \\ \frac{dS}{d\tau} &= -bS + cHS \end{aligned} \tag{1}$$

1. This is a coupled two-dimensional dynamical system with continuous time. What kind of attractors can occur in such system.

2. Show that the system (1) with $x = \alpha H$, $y = \alpha S$ and $t = \beta\tau$ can be transformed onto a form determined by only one parameter B

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} &= x - xy \\ \frac{dy}{dt} &= \dot{y} &= -By + xy \end{aligned} \tag{2}$$

   Find the parameters α, β, B in terms of a, b and c.

3. Show that equations (2) imply that $K = x + y - B\ln|x| - \ln|y|$ is conserved.

4. Find the fixed point of (2) and analyze them in term of stability (we assume $B > 0$).

5. Sketch the phase plane of the system for both positive and negative values of x and y (e.g. nullclines, eigendirections, approximate flow lines). Use for instance the value $B = 1$ in the sketch.

II.

Consider the van der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{3}$$

with $\epsilon < 0$. Let us study the phase plane (x, y) with $\dot{x} = y$.

6. Derive an equation for $\dot{r}$ where $r^2 = x^2 + y^2$.

7. Show that there exist "trapping regions" defined by circles $x^2 + y^2 = r^2$ of radii $r \in [0; r_m[$ such that all solutions starting from initial conditions inside these circles tend to the origin.

8. Argue for and determine a maximal value of r_m.

III.

Consider the discrete mapping

$$x_{n+1} = f(x_n) = 1 - \mu x_n^2 \tag{4}$$

where we assume $0 < \mu < 2$, and with an initial value x_0 which satisfy $|x_0| < 1$.

9. Determine the fixed points of (4). What is the largest value of $\mu = \mu_1$ that corresponds to a stable fixed point.

10. The period two cycle iterates between two values x_+ and x_-. Show that $x_+ + x_- = 1/\mu$

11. Use this result to show that $x_\pm = \frac{1 \pm \sqrt{4\mu - 3}}{2\mu}$.

12. For which value of μ is the two-cycle superstable.

13. Now let $\mu = \frac{7}{4}$. With the initial value, $x_0 = \frac{6}{7}$, do you obtain a two-cycle ? Is it stable ?

14. What is the largest value μ_2 for which the two-cycle is stable.