Summary
Dynamical systems and chaos
Spring 2014

One 1st order ODE

$$\dot{x} = f(x)$$

Potential V(x):

$$f(x) = -\left(\frac{dv}{dx}\right)$$

Bifurcations: $\dot{x} = f(x, r)$

-find for which r when fix points are created
- How is stability changed when r is changed
-draw x*(r) and mark stability

Fix points (Fp:s), x*:

$$f(x^*) = 0$$

Stability:

$$f'(x^*) < 0 \Rightarrow stable$$
$$f'(x^*) > 0 \Rightarrow unstable$$
$$f'(x^*) = 0 \Rightarrow \text{check } graphically$$

Saddle-node: Fp:s are created and destroyed

Transcritical: There is always and fp and is never destroyed but stability may change.

Pitchfork(sub and super):
See pp 56 and pp 59

TWO 1^{st} order ODEs

Stability: (pp 137)

Saddle: $\Delta < 0$

Nodes: $\tau^2 - 4\Delta > 0$

$\tau < 0$ stable

$\tau > 0$ Unstable

Spirals $\tau^2 - 4\Delta < 0$

$\tau < 0$ Stable

$\tau > 0$ Unstable

Centers $\tau = 0$ and $\Delta > 0$

Borderline cases $\Delta = 0$

Nullclines:

Look at $\dot{y}$ When $\dot{x} = f(x, y) = 0$

Look at $\dot{x}$ When $\dot{y} = g(x, y) = 0$

Two-dimensional linear systems:

(pp 130)

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, x = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\mathbf{x} = A\,\mathbf{x}$$

$$\tau = trace(A) = a + d$$
$$\Delta = deta(A) = ad - bc$$
$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

Two-dimensional nonlinear systems: (pp 150-151)

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

Fp: (x^*, y^*)

$$f(x^*, y^*) = 0, g(x^*, y^*) = 0$$

Stability: Use the Jacobian and the same analysis as for linear systems!
EXCEPTION: Centers and borderline cases.
Additional analysis has do be done.

$$A = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix} (x^*, y^*)$$

Bifurcation:
Saddle node, transcritical and pitchfork (pp 241-247)
Hopf bifurcation:
-check when the eigenvalues are imaginary
$\tau - 4\Delta < 0$
-check when they change stability, i.e. when τ change sign

Supercritical: Stable spiral changes into an unstable spiral surrounded by a small nearly elliptic limit cycle.
Subcritical: (pp 251-252)

Limit cycles:
Isolated closed trajectory

Rule out:
Gradient systems pp 199
Liapunov functions pp 201

Establish:
Pointcare-Bendixson theorem and trapping regions pp 203

Nonlinear centers:

Nonlinear centers for conserved systems (theorem 6.5.1 at page 163)

Nonlinear systems for reversible systems: (theorem 6.6.1 at page 164)

Index theory:
Global information about the stability (page 174)

Averaged Equations:

(pp 223-224)

$$\ddot{x}+x+\epsilon h(x,\dot{x})=0$$
$$\tau=t, T=\epsilon t$$
$$x_0=r(T)\cos(\tau+\phi(T))$$
$$\theta=\tau+\phi$$
$$r'=<h\sin(\theta)>$$
$$r\ \phi=<h\cos(\theta)>$$

Chaos:

(from page 301)

Lorenz equations

(from page 311)

1D-maps: $x_{n+1}=f(x_n)$

Fps:

x* such as $f(x^*)=x^*$

Stability:

Stable if $|f(x^*)|<1$

2-cycles:

Two points p and q such that f(p)=q and f(q)=p. So f(f(p))=p

$f^2(x)=f(f(x))$

Stability:

Stable if $|f(q)f(p)|<1$

3-cycles, 4-cycles use the same reasoning

Fractals:

Similarity dimension:

m=number of copies

r=scale factor

$$d=\frac{\ln m}{\ln r}$$

Box dimension: (page 409)