# Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 12

March 11, 2015

# 8 Bifurcations Revisited

## 8.4 Global Bifurcations of Cycles

### 8.4.5

Denote $\theta = t + \phi$, and we have

$$\begin{aligned} h =& bx^3 + k\dot{x} - ax - F\cos t \\ =& b(r\cos\theta)^3 + k(-r\sin\theta) - a(r\cos\theta) - F\cos(\theta - \phi) \\ =& br^3\cos^3\theta - kr\sin\theta - ar\cos\theta - F\cos\theta\cos\phi - F\sin\theta\sin\phi \end{aligned}$$

Therefore

$$\begin{aligned} r' =& \langle h\sin\theta\rangle \\ =& br^3\langle\cos^3\theta\sin\theta\rangle - kr\langle\sin^2\theta\rangle - ar\langle\sin\theta\cos\theta\rangle - F\cos\phi\langle\cos\theta\sin\theta\rangle - F\sin\phi\langle\sin^2\theta\rangle \\ =& -\frac{1}{2}kr - \frac{1}{2}F\sin\phi = -\frac{1}{2}(kr + F\sin\phi) \end{aligned}$$

and

$$\begin{aligned} r\theta' =& \langle h\cos\theta\rangle \\ =& br^3\langle\cos^4\theta\rangle - kr\langle\sin\theta\cos\theta\rangle - ar\langle\cos^2\theta\rangle - F\cos\phi\langle\cos^2\theta\rangle - F\sin\phi\langle\sin\theta\cos\theta\rangle \\ =& \frac{3}{8}br^3 - \frac{1}{2}ar - \frac{1}{2}F\cos\phi \\ =& -\frac{1}{8}(4ar + 4F\cos\phi - 3br^3) \\ \theta' =& -\frac{1}{8}(4a - 3br^2 + \frac{4F}{r}\cos\phi) \end{aligned}$$

# 10 One-Dimensional Maps

## 10.4 Periodic Windows

### 10.4.3

Solving $f'(x^*) = -2rx^* = 0$ gives $x^* = 0$. Hence, 0 is a solution for $f^3(x) = x$, so

$$\begin{aligned} &f^3(x) = f(f(f(x))) = f(f(1 - rx^2)) = f(1 - r(1 - rx^2)^2) = 1 - r[1 - r(1 - rx^2)^2]^2 \\ \rightarrow &f^3(0) = 1 - r(1 - r)^2 = 1 - r + 2r^2 - r^3 = 0 \end{aligned}$$

## 10.5 Lyapunov Exponent

### 10.5.1

The solution for $x_n$ given an initial value $x_0$ is $x_n = r^n x_0$. Therefore,

$$\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{r^n(x_0 + \delta_0) - r^n x_0}{\delta_0} \right| = \frac{1}{n} \ln |r^n| = \ln |r|$$

# 11 Fractals

## 11.2 Cantor Set

### 11.2.1

Since each time we remove 1/3 of available length, the length after $n$ removal is $l_n = (2/3) \cdot (2/3) \cdots (2/3) = (2/3)^n$. Therefore, the $(n+1)$-th removal removes a length of $r_{n+1} = l_n(1/3) = 2^n/(3^{n+1})$. Hence, the total removal is

$$R = \sum_{n=1}^{\infty} r_n = \frac{1}{3} \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n-1}} = \frac{1}{3} \frac{1}{1 - (2/3)} = 1$$

Therefore, the measure of cantor set is 0.