Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 10

March 9, 2015

9 Lorenz Equations

9.1 A Chaotic Waterwheel

9.1.4

(a) $\dot{P} = 0$ gives $ED = P$ and $\dot{D} = 0$ gives $D = \lambda + 1 - \lambda EP$. Therefore

$$P = \frac{(\lambda+1)E}{\lambda E^2 + 1}$$

and

$$\dot{E} = \kappa \left(\frac{(\lambda+1)E}{\lambda E^2 + 1} - E \right)$$

Denote the terms on the right hand side as $f(E)$, and

$$f'(E) = \kappa \left[\frac{(\lambda+1)(1-\lambda E^2)}{(\lambda E^2+1)^2} - 1 \right]$$

- If $\lambda < 0$, there are 3 fix points $E_1^* = 0$, $E_2^* = 1$ and $E_3^* = -1$. The derivate at $E_1^* = 0$ is $f'(E_1^*) = \lambda\kappa < 0$, so it is stable.
- If $\lambda = 0$, there are infinite fix points. The derivative at $E^* = 0$ is $f'(E^*) = 0$, showing that $\lambda = 0$ is the bifurcation point.
- If $\lambda > 0$, there are 3 fix points $E_1^* = 0$, $E_2^* = 1$ and $E_3^* = -1$. The derivate at $E_1^* = 0$ is $f'(E_1^*) = \lambda\kappa > 0$, so it is unstable.

Therefore, it is a degenerate pitchfork bifurcation.

(b) By comparing the equation of $\dot{\tilde{x}}$ and $\dot{E}$, we have $\tilde{t} = \kappa t/\sigma$, $\tilde{x} = \alpha E$ and $\tilde{y} = \alpha P$ where α needs to be determined. For verification of such transformation, we have

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} = \frac{\alpha \mathrm{d}E}{\kappa/\sigma \mathrm{d}t} = \frac{\alpha\sigma}{\kappa}\frac{\mathrm{d}E}{\mathrm{d}t} = \alpha\sigma(P - E) = \sigma(\tilde{y} - \tilde{x})$$

Then we compute $\mathrm{d}\tilde{y}/\mathrm{d}\tilde{t}$ with $\dot{P}$ to obtain the transformation of γ_1 and D.

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}} = \frac{\alpha\sigma \mathrm{d}P}{\kappa \mathrm{d}t} = \frac{\alpha\sigma}{\kappa}\gamma_1(ED - P) = \frac{\sigma}{\kappa}\gamma_1(\tilde{x}D - \tilde{y}) = \tilde{x}(r - \tilde{z}) - \tilde{y}$$

Therefore $\gamma_1 = \kappa/\sigma$ and $D = r - \tilde{z}$.

Finally compute $\mathrm{d}\tilde{z}/\mathrm{d}\tilde{t}$ with $\dot{E}$ to determine the transformation of the remaining variables.

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{t}} = -\frac{\mathrm{d}D}{\kappa/\sigma \mathrm{d}t} = -\frac{\sigma}{\kappa}\frac{\mathrm{d}D}{\mathrm{d}t} = -\frac{\sigma\gamma_2}{\kappa}(\lambda+1-D-\lambda EP) = -\frac{\sigma\gamma_2}{\kappa}(\lambda+1-r) - \frac{\sigma\gamma_2}{\kappa}\tilde{z} + \frac{\sigma\gamma_2}{\kappa}\alpha^2\lambda\tilde{x}\tilde{y} = \tilde{x}\tilde{y} - b\tilde{z}$$

Therefore $\lambda = r-1$, $\gamma_2 = b\kappa/\sigma$ with a special relationship $\alpha = [b(r-1)]^{-1/2}$.

9.2 Simple Properties of the Lorenz Equations

9.2.1

(a) The two fix points are $(x^*, y^*, z^*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$, if $r > 1$, and the Jacobian is

$$\mathbf{J}|_{(x^*,y^*,z^*)} = \left[\begin{array}{ccc} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{array}\right]\Bigg|_{(x^*,y^*,z^*)} = \left[\begin{array}{ccc} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{array}\right]$$

The characteristic polynomial is

$$\begin{aligned} P(\lambda) = |\lambda\mathbf{I} - \mathbf{J}| = \det &\left[\begin{array}{ccc} \lambda+\sigma & -\sigma & 0 \\ -1 & \lambda+1 & \pm\sqrt{b(r-1)} \\ \mp\sqrt{b(r-1)} & \mp\sqrt{b(r-1)} & \lambda+b \end{array}\right] \\ =&(\lambda+\sigma)(\lambda+1)(\lambda+b) + \sigma b(r-1) - \sigma(\lambda+b) + (\lambda+\sigma)b(r-1) \\ =&\lambda^3 + (\sigma+b+1)\lambda^2 + b(\sigma+r)\lambda + 2\sigma b(r-1) = 0 \end{aligned}$$

(b) Denote the solutions of $P(\lambda) = 0$ as $\lambda_{1,2,3}$ and we have

$$\lambda_1 + \lambda_2 + \lambda_3 = -(\sigma+b+1), \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = b(\sigma+r), \lambda_1\lambda_2\lambda_3 = -2\sigma b(r-1)$$

Assume $\lambda_1 = \omega i$ and $\lambda_2 = -\omega i$ where $\omega \in \mathbb{R}$, and the first equation gives $\lambda_3 = -(\sigma+b+1)$ and the second equation gives $\omega = \sqrt{b(\sigma+r)}$. Solving the third equation gives that

$$-(\sigma+b+1)b(\sigma+r) = -2\sigma b(r-1) \to r = \frac{(\sigma+b+1)b\sigma + 2\sigma b}{2\sigma b - (\sigma+b+1)b} = \sigma\frac{\sigma+b+3}{\sigma-b-1}$$

Since $r_H > 0$ and $\sigma, b > 0$, we should have $\sigma > b+1$.

(c) $\lambda_3 = -(\sigma+b+1)$.

9.2.6

(a) The divergence of the vector field is

$$\begin{aligned} \nabla\cdot\vec{f} =&\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ =& -v - v + 0 = -2v \end{aligned}$$

Therefore the volume V follows the relationship that $\dot{V} = -2vV$ and $V = V(0)e^{-2vt}$. V is decreasing, so the system is dissipative.

(b) Note that $x^* = \pm k$ and $y^* = \pm k^{-1}$, obviously $x^*y^* = (\pm k)(\pm k^{-1}) = 1$ and $\dot{z} = 0$ is satisfied.

Since $z = vk^2$, we have $zy - vx = (vk^2)(\pm k^{-1}) - v(\pm k) = (\pm vk) - (\pm vk) = 0$ and $\dot{x} = 0$ is satisfied.

Finally by setting $\dot{y} = 0$, we have

$$vy = (z-a)x \to \pm vk^{-1} = \pm(vk^2 - a)k \to vk^{-1} = (vk^2 - a)k \to vk^{-2} = vk^2 - a \to v(k^2 - k^{-2}) = a$$

(c) Without loss of generality, the Jacobian at the fix point (k, k^{-1}, vk^2) is

$$\mathbf{J}|_{(k,k^{-1},vk^2)} = \left[\begin{array}{ccc} -v & z & y \\ z-a & -v & x \\ -y & -x & 0 \end{array}\right]\Bigg|_{(k,k^{-1},vk^2)} = \left[\begin{array}{ccc} -v & vk^2 & k^{-1} \\ vk^2 - a & -v & k \\ -k^{-1} & -k & 0 \end{array}\right]$$

so the eigenvalues of the Jacobian follows

$$0 = \det(\mathbf{J}|_{(k,k^{-1},vk^2)} - \lambda\mathbf{I}) = \det\left[\begin{array}{ccc} -v-\lambda & vk^2 & k^{-1} \\ vk^2 - a & -v-\lambda & k \\ -k^{-1} & -k & -\lambda \end{array}\right]$$

and we have

$$\lambda^3 + 2v\lambda^2 + (k^2 + k^{-2})\lambda + 2v(k^2 + k^{-2}) = 0$$

Since $v, k^2 > 0$, we have

$$\lambda_1 + \lambda_2 + \lambda_3 = -2v < 0; \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = k^2 + k^{-2} > 0 \text{ and } \lambda_1\lambda_2\lambda_3 = -2v(k^2 + k^{-2}) < 0$$

The third relation only yields two possible cases: (1) $\lambda_1 < 0, \lambda_{2,3} > 0$ and (2) $\lambda_{1,2,3} < 0$. However, for case (1), we have

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \lambda_1(\lambda_2 + \lambda_3) + \lambda_2\lambda_3 < -(\lambda_2 + \lambda_3)^2 + \lambda_2\lambda_3 = -\lambda_2^2 - \lambda_2\lambda_3 - \lambda_3^2 < 0$$

contradicting to the second relation. Therefore, all three eigenvalues are negative and the fix point is stable.

9.4 Lorenz Map

9.4.2

(a) $f(x)$ likes a tent.

(b)

- If $0 \le x^* \le 1/2$, solving $f(x^*) = x^*$ gives $x^* = 0$. It is unstable since $f'(x^*) = 2 > 1$.
- If $1/2 < x^* \le 1$, solving $f(x^*) = x^*$ gives $x^* = 2/3$. It is unstable since $f'(x^*) = -2 < -1$.

(c)

- If $0 \le x^* \le 1/2$ and $0 \le f(x^*) \le 1/2$, $f(f(x^*)) = f(2x^*) = 4x^* = x^*$. So $x^* = 0$ and it is not a period-2 orbit.

- If $0 \leq x^* \leq 1/2$ and $1/2 < f(x^*) \leq 1$ (and same for $1/2 < x^* \leq 1$ and $0 \leq f(x^*) \leq 1/2$), $f(f(x^*)) = f(2x^*) = 2 - 4x^* = x^*$. So $x^* = 2/5$, $f(x^*) = 4/5$ and it is unstable.

- If $1/2 < x^* \leq 1$ and $1/2 < f(x^*) \leq 1$, $f(f(x^*)) = f(2 - 2x^*) = 4x^* - 2 = x^*$. So $x^* = 2/3$ and it is not a period-2 orbit.

(d) Period-3. For simplicity, here describe the relationship between x^*, $f(x^*)$, $f(f(x^*))$ and $1/2$, respectively.

- $<,<,<$: $f(f(f(x^*))) = f(f(2x^*)) = f(4x^*) = 8x^* = x^*$. So $x^* = 0$ and it is not a period-3 orbit.

- $<,<,>$ (same for $<,>,<$ and $>,<,<$): $f(f(f(x^*))) = f(4x^*) = 2 - 8x^* = x^*$. So $x^* = 2/9, f(x^*) = 4/9, f(f(x^*)) = 8/9$. It is unstable.

- $<,>,>$ (same for $>,<,>$ and $>,>,<$): $f(f(f(x^*))) = f(f(2x^*)) = f(2-4x^*) = 8x^*-2 = x^*$. So $x^* = 2/7, f(x^*) = 4/7, f(f(x^*)) = 6/7$ and it is unstable.

- $>,>,>$: $f(f(f(x^*))) = f(f(2 - 2x^*)) = f(4x^* - 2) = 6 - 8x^* = x^*$. So $x^* = 2/3$ and it is not a period-3 orbit.

Period-4.

- $<,<,<,<$: $f(f(f(f(x^*)))) = 16x^* = x^*$, so $x^* = 0$ and it is not a period-4 orbit.

- $<,<,<,>$ (and 3 others): $f(f(f(f(x^*)))) = f(8x^*) = 2 - 16x^* = x^*$. So $x^* = 2/17, f(x^*) = 4/17, f(f(x^*)) = 8/17, f(f(f(x^*))) = 16/17$. It is unstable.

- $<,<,>,>$ (and 3 others): $f(f(f(f(x^*)))) = f(f(4x^*)) = f(2 - 8x^*) = 16x^* - 2 = x^*$. So $x^* = 2/15, f(x^*) = 4/15, f(f(x^*)) = 8/15, f(f(f(x^*))) = 14/15$. It is unstable.

- $<,>,<,>$ (and 1 other): $f(f(f(f(x^*)))) = f(f(2 - 4x^*)) = 16x^* - 6 = x^*$. So $x^* = 2/5, f(x^*) = 4/5$ and it is not a period-4 orbit.

- $<,>,>,>$ (and 3 others): $f(f(f(f(x^*)))) = f(f(f(2x^*))) = f(f(2 - 4x^*)) = f(8x^* - 2) = 6 - 16x^* = x^*$. So $x^* = 6/17, f(x^*) = 12/17, f(f(x^*)) = 10/17, f(f(f(x^*))) = 14/17$. It is unstable.

- $>,>,>,>$: $f(f(f(f(x^*)))) = f(f(f(2-2x^*))) = f(f(4x^*-2)) = f(6-8x^*) = 16x^*-10 = x^*$. So $x^* = 2/3$ and it is not a period-4 orbit.