Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 09

March 4, 2015

8 Bifurcations Revisited

8.2 Hopf Bifurcations

8.2.1

Rewrite the 2nd order ODE as

$$\dot{x} = y, \dot{y} = -\mu(x^2 - 1)y - x + a$$

and the fix point is $(a, 0)$. Linearize the system around the fix point and the Jacobian is

$$\mathbf{J}|_{(a,0)} = \left[\begin{array}{cc} 0 & 1 \\ -2\mu xy - 1 & -\mu(x^2-1) \end{array}\right]\Bigg|_{(a,0)} = \left[\begin{array}{cc} 0 & 1 \\ -1 & -\mu(a^2-1) \end{array}\right]$$

and the eigenvalues are

$$\lambda_{1,2} = \frac{-\mu(a^2-1) \pm \sqrt{\mu^2(a^2-1)^2 - 4}}{2}$$

Hopf bifurcation requires that $-\mu(a^2 - 1) = 0$ and $\mu^2(a^2 - 1)^2 - 4 < 0$, and the solution is $a = \pm 1$ or $\mu = 0$. However, with $\mu = 0$, the system is a linear one and we have a linear center. Therefore the solution is $a = \pm 1$.

8.2.9

(a) The nullclines are

$$\dot{x} = 0 \rightarrow x = 0 \text{ or } y = (b - x)(1 + x)$$
$$\dot{y} = 0 \rightarrow y = 0 \text{ or } y = \frac{x}{a(1+x)}$$

(b) The intersections between the nullclines are

- Between $x = 0$ and $y = 0$, $(0, 0)$
- Between $x = 0$ and $y = x/a(1 + x)$, $(0, 0)$
- Between $y = (b - x)(1 + x)$ and $y = 0$, $(b, 0)$

- Between $y=(b-x)(1+x)$ and $y=x/a(1+x)$, solving the equation yields the following cubic polynomial
$$x^3+(2-b)x^2+(1+a-2b)x-b=0$$
Clearly, the 3 solutions $(x_{1,2,3})$ satisfy that $x_1x_2x_3=b>0$. Therefore, at least one of them is positive, indicating that at least one positive fix point $(x^*,y^*)\in\mathbb{R}^2_{>0}$ exists.

(c) The Jacobian at the fix point is
$$\mathbf{J}|_{(x^*,y^*)}=\begin{bmatrix} b-2x^*-\frac{y^*}{(x^*+1)^2} & -\frac{x^*}{1+x^*} \\ \frac{y}{(x^*+1)^2} & \frac{x^*}{1+x^*}-2ay^* \end{bmatrix}=\begin{bmatrix} \frac{b-x^*}{x^*+1}x^*-x^* & -\frac{x^*}{1+x^*} \\ \frac{b-x^*}{x^*+1} & -\frac{x^*}{1+x^*} \end{bmatrix}$$
The Hopf bifurcation requires that the trace to be 0 and the determinant to be positive. The trace is
$$\mathrm{tr}\mathbf{J}|_{(x^*,y^*)}=\frac{b-x^*}{x^*+1}x^*-x^*-\frac{x^*}{1+x^*}=\frac{x^*}{1+x^*}(b-2x^*-2)$$
$\mathrm{tr}\mathbf{J}|_{(x^*,y^*)}=0$ yields $x^*=(b-2)/2$. The determinant is
$$\det\mathbf{J}|_{(x^*,y^*),x^*=(b-2)/2}=-\left(\frac{b-x^*}{x^*+1}x^*-x^*\right)\frac{x^*}{1+x^*}+\frac{b-x^*}{x^*+1}\frac{x^*}{1+x^*}=\frac{2x^*}{1+x^*}>0$$
Therefore, there exists a Hopf bifurcation at the fix point. We then need to derive a_c as follow
$$a_c=\frac{x^*}{(b-x^*)(1+x^*)^2}=\frac{(b-2)/2}{[(b+2)/2](b/2)^2}=\frac{4(b-2)}{(b+2)b^2}$$

(d)

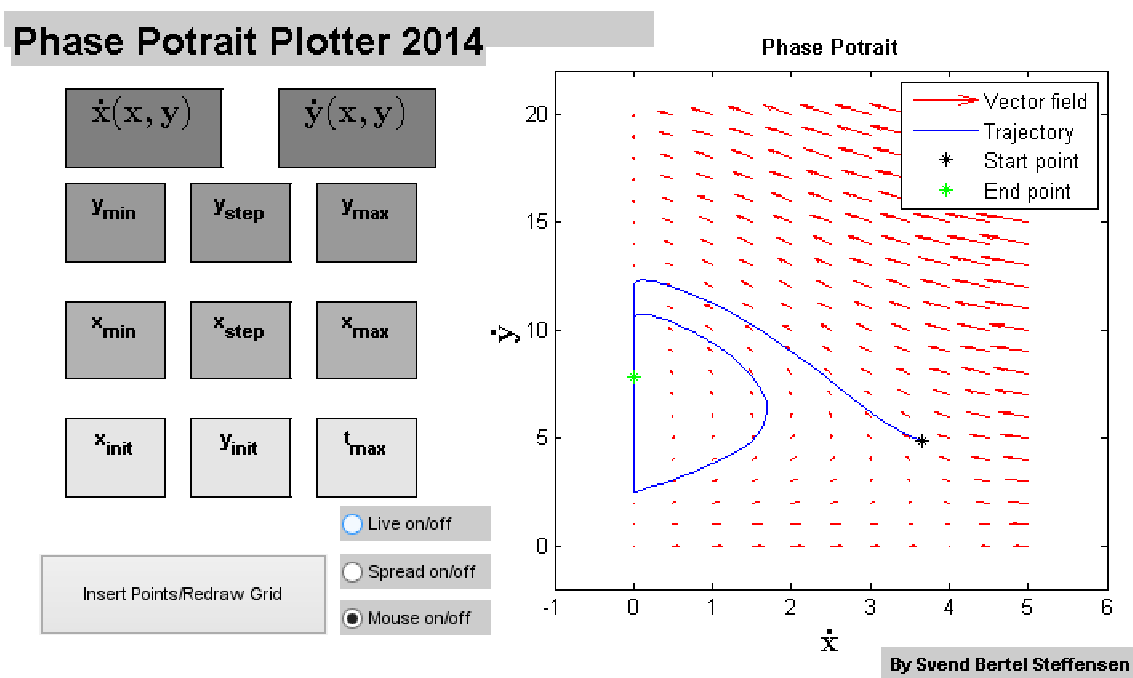

Figure 1: 8.2.9. $a=1/120, b=4$

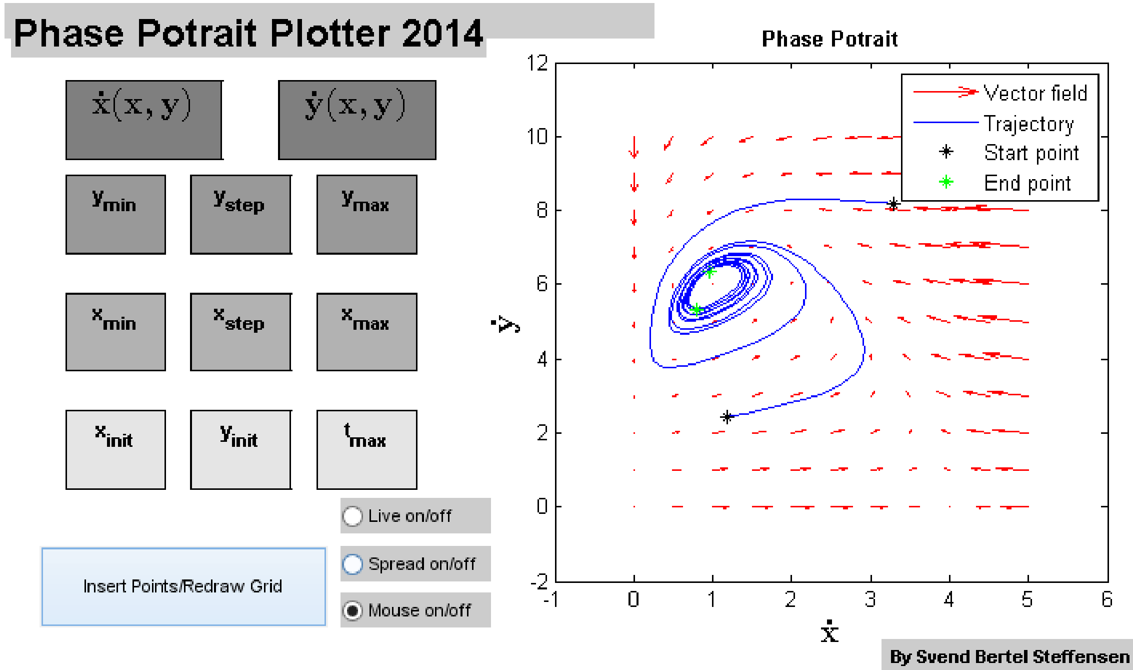

Figure 2: 8.2.9. $a = 1/12, b = 4$

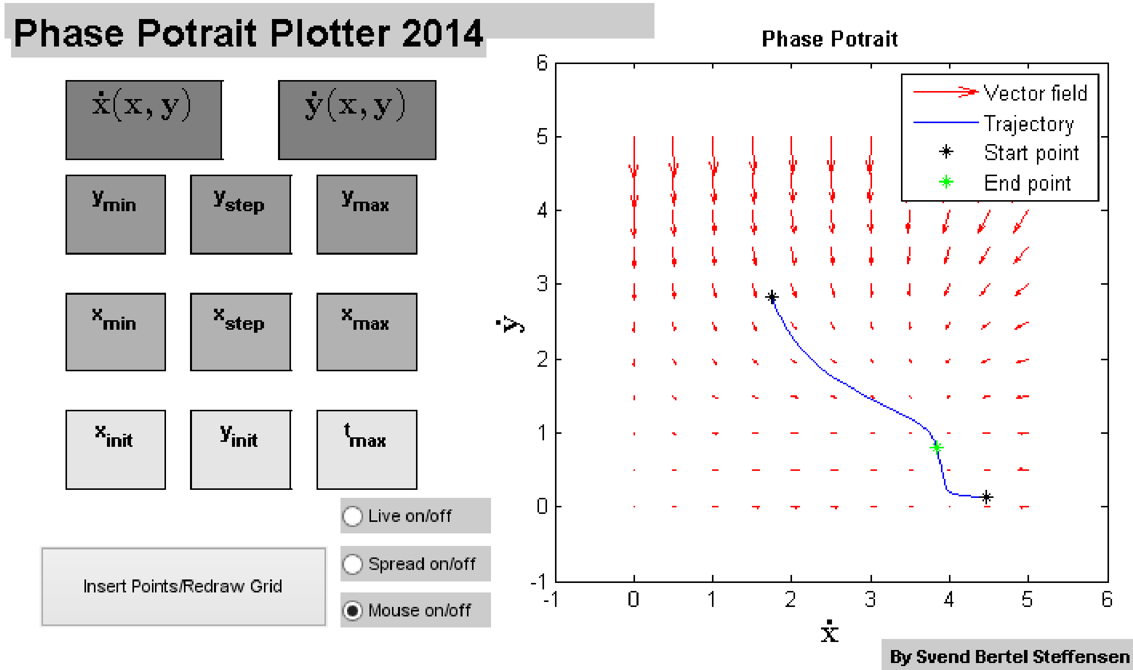

Figure 3: 8.2.9. $a = 1, b = 4$

8.3 Oscillating Chemical Reactions

8.3.1

(a) The fix point is $(1, b/a)$, and the Jacobian is

$$\mathbf{J}|_{(1,b/a)} = \begin{bmatrix} -b-1-2axy & ax^2 \\ b-2axy & -ax^2 \end{bmatrix}\Bigg|_{(1,b/a)} = \begin{bmatrix} b-a & a \\ -b & -a \end{bmatrix}$$

and the eigenvalues are

$$\lambda_{1,2} = \frac{-(a+1-b) \pm \sqrt{(a+1-b)^2 - 4a}}{2}$$

- If $a + 1 - b > 0$, or $b < a + 1$, the fix point is stable;
- If $b > a + 1$, the fix point is unstable;
  - If $(a + 1 - b)^2 - 4a > 0$, it is a saddle;
  - If $(a + 1 - b)^2 - 4a < 0$, it is an unstable spiral
- If $b = a + 1$, the fix point is a center.

(c) Denote $b_c = a + 1$. If $b > b_c$, the fix point is unstable spiral. Together with the fact that there is a trapping region, there must exist a closed orbits, i.e. limit cycle.

(d) $b > b_c$.

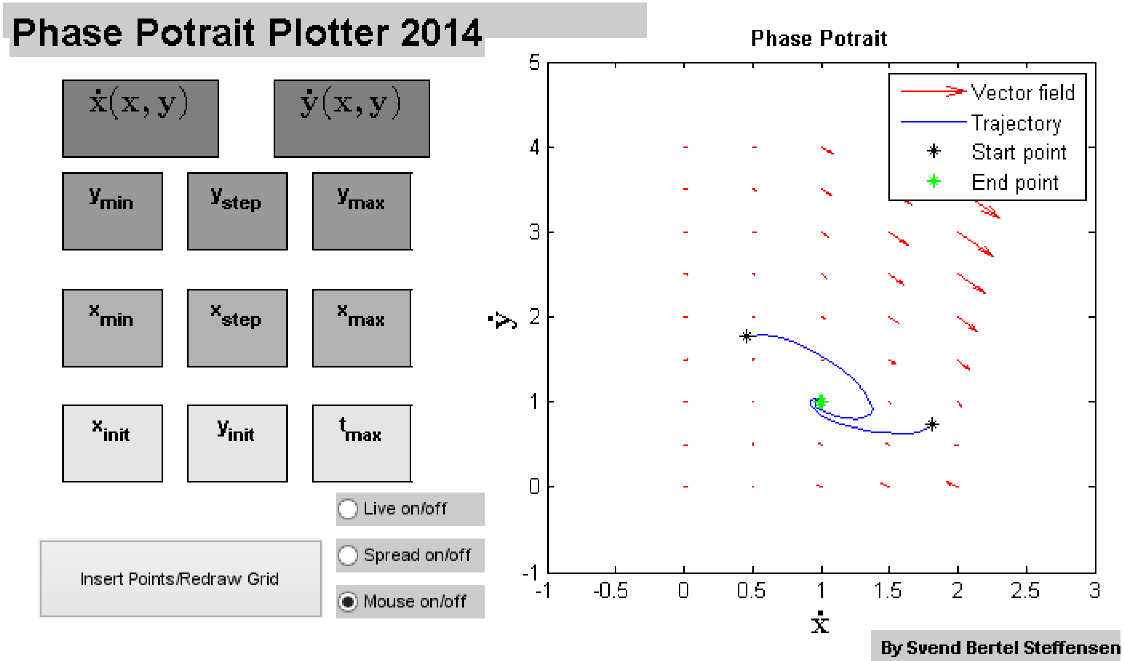

Figure 4: 8.2.9. $a = 1, b = 1$

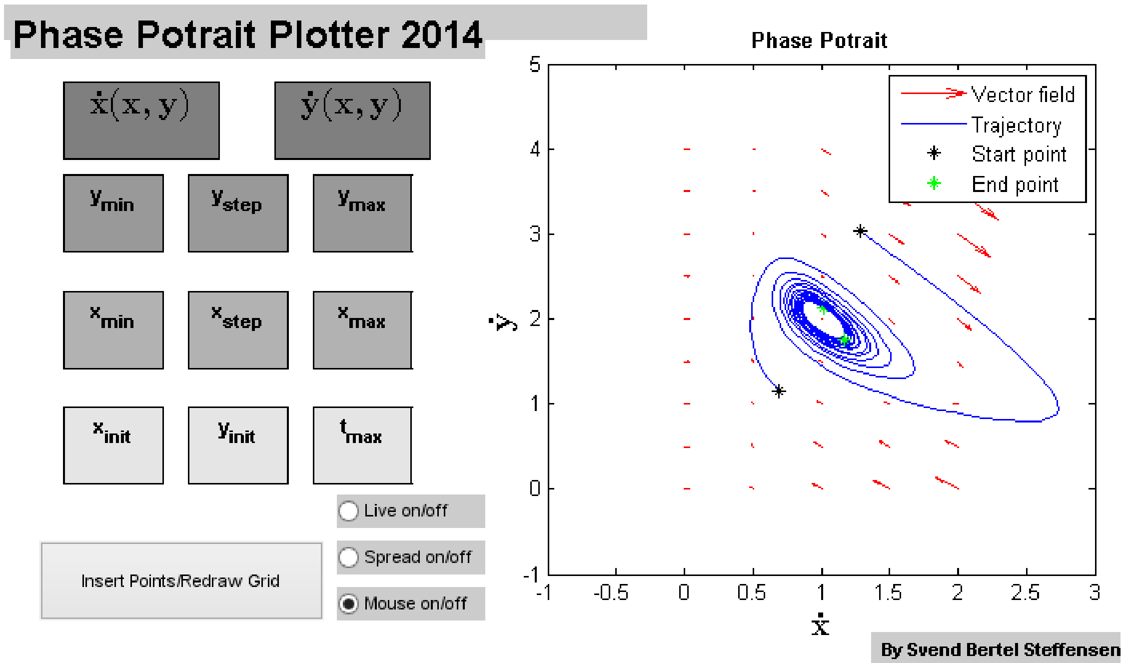

Figure 5: 8.2.9. $a = 1, b = 2$

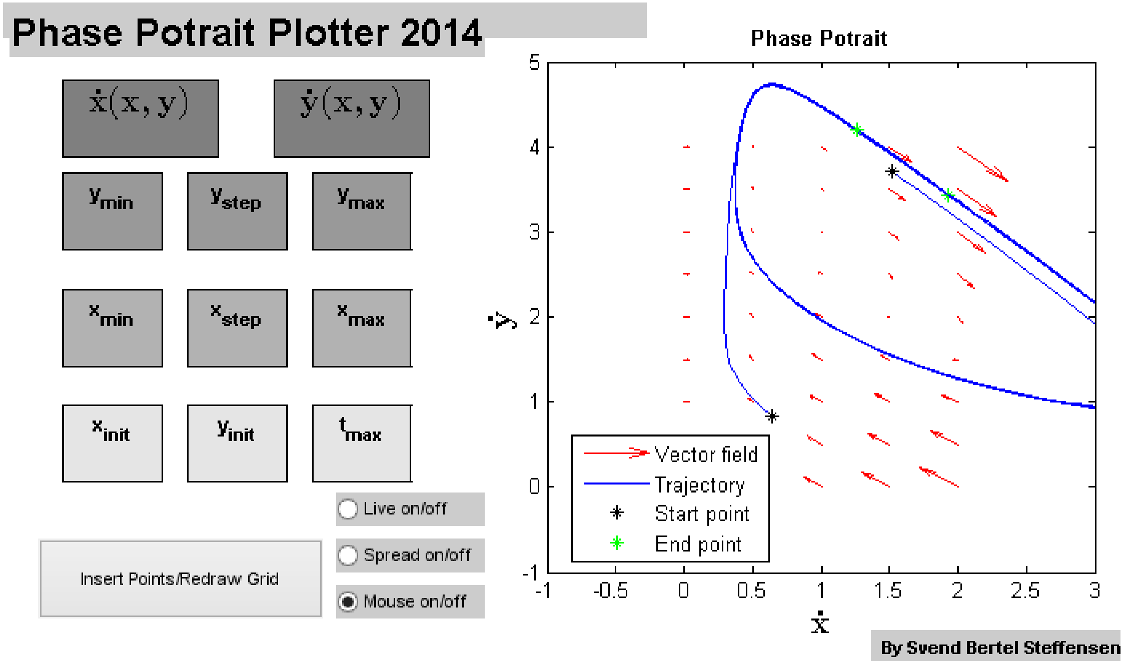

Figure 6: 8.2.9. $a = 1, b = 3$

(e) The frequency is

$$\omega = \frac{\sqrt{-(a+1-b)^2 + 4a}}{2}\Bigg|_{b \approx b_c} = \sqrt{a}$$

Therefore the period is $T = 2\pi/\omega = 2\pi/\sqrt{a}$.

8.7 Poincare Maps

8.7.1

$$\begin{aligned}\int_{r_0}^{r_1} \frac{\mathrm{d}r}{r(1-r^2)} &= \int_{r_0}^{r_1} \left(\frac{1}{r} - \frac{1}{2}\frac{1}{r-1} - \frac{1}{2}\frac{1}{r+1}\right) \mathrm{d}r \\ &= \int_{r_0}^{r_1} \mathrm{d}\ln \frac{|r|}{\sqrt{|r^2-1|}} \\ &= \ln \frac{|r_1|}{\sqrt{|r_1^2-1|}} - \ln \frac{|r_0|}{\sqrt{|r_0^2-1|}}\end{aligned}$$

Since $0 < r_0, r_1 < 1$, we have

$$\int_{r_0}^{r_1} \frac{\mathrm{d}r}{r(1-r^2)} = \ln \frac{r_1}{\sqrt{1-r_1^2}} - \ln \frac{r_0}{\sqrt{1-r_0^2}} = \ln \frac{r_1}{r_0} \frac{\sqrt{1-r_0^2}}{\sqrt{1-r_1^2}}$$

Therefore

$$\int_{r_0}^{r_1} \frac{\mathrm{d}r}{r(1-r^2)} = 2\pi \to \frac{r_1^2}{1-r_1^2} = \mathrm{e}^{4\pi} \frac{r_0^2}{1-r_0^2} \to r_1^2 = \frac{\mathrm{e}^{4\pi} r_0^2/(1-r_0^2)}{1+\mathrm{e}^{4\pi} r_0^2/(1-r_0^2)} \to r_1 = \frac{1}{\sqrt{1+\mathrm{e}^{-4\pi}(r_0^{-2}-1)}}$$

It can be shown that when $r_0, r_1 > 1$, the expression for $P(r)$ is identical.

The derivative of $P(r)$ is

$$P'(r) = \frac{\mathrm{e}^{-4\pi}/r^3}{\left(1+\mathrm{e}^{-4\pi}(r^{-2}-1)\right)^{3/2}}$$

and

$$P'(r^*) = \frac{\mathrm{e}^{-4\pi}/r^{*3}}{\left(1+\mathrm{e}^{-4\pi}(r^{*-2}-1)\right)^{3/2}} = \mathrm{e}^{-4\pi}$$