Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 08

February 25, 2015

7 Limit Cycles

7.3 Poincare-Bendixson Theorem

7.3.4

(c) Consider another two ellipses with different sizes. The first one is $4x^2 + y^2 = 4$ and the second one is $4x^2 + y^2 = 1/4$. The area between the two ellipses is the region R we want to study. The objective is to show that R is a trapping region.

Firstly consider $4x^2 + y^2 = 4$ (the outer ellipse). Denote $x = r\cos\theta$ and $y = 2r\sin\theta$. Changing to polar coordinate gives

$$4x^2 + y^2 = 4r^2 \to 8x\dot{x} + 2y\dot{y} = 8r\dot{r} \to 8r^2(1-4r^2) = 8r\dot{r} \to \dot{r} = r - 4r^3$$

A point on the ellipse satisfies that $r^* = 1$. The derivative is $f'(r^*) = 1 - 12r^2|_{r^*} = -11 < 0$. Therefore, the points on the outer ellipse are attracted inside.

Then consider $4x^2 + y^2 = 1/4$ (the inner ellipse). Use the same technique and denote $x = r\cos\theta/4$ and $y = r\sin\theta/2$. Changing to polar coordinate gives

$$4x^2 + y^2 = \frac{1}{4}r^2 \to \dot{r} = r - \frac{1}{4}r^3$$

The derivative at $r^* = 1$ is $1/4$, so all the points at the inner ellipse is repelled. Therefore, R is a trapping region and there exists a closed orbit inside R.

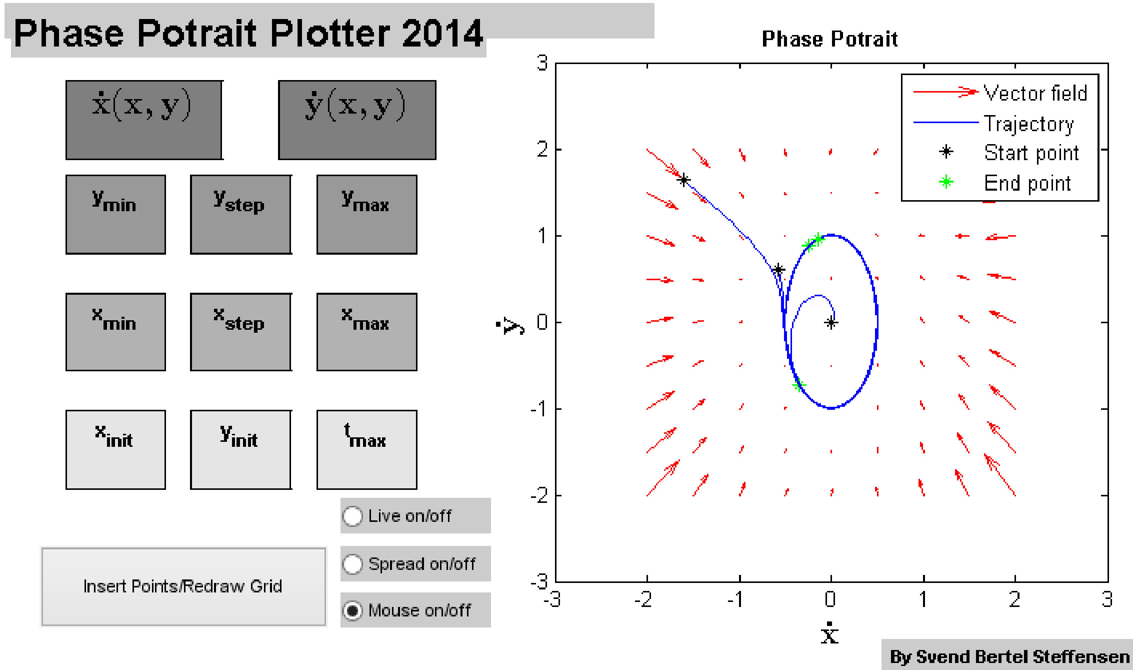

Figure 1: 7.3.4

7.6 Weakly Nonlinear Oscillators

7.6.5

Given $x_0 = r(T)\cos(\tau + \phi(T))$, we have

$$h\left(x_0, \frac{\partial x_0}{\partial \tau}\right) = h(r(T)\cos(\tau+\phi(T)), -r(T)\sin(\tau+\phi(T))) = r(T)^3\cos(\tau+\phi(T))\sin^2(\tau+\phi(T))$$

Therefore, the average equation for $r(T)$ is

$$\frac{\mathrm{d}r(T)}{\mathrm{d}T} = \langle h\sin(\tau+\phi(T))\rangle = r(T)^3\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\sin^3\theta\mathrm{d}\theta = \frac{r(T)^3}{8}\sin^4\theta|_0^{2\pi} = 0$$

and the equation for $\phi(T)$ is

$$r(T)\frac{\mathrm{d}\phi(T)}{\mathrm{d}T} = \langle h\cos(\tau+\phi(T))\rangle = r(T)^3\frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\sin^2\theta\mathrm{d}\theta = \frac{r(T)^3}{8} \to \frac{\mathrm{d}\phi(T)}{\mathrm{d}T} = \frac{r(T)^2}{8}$$

Therefore, $r(T) = r_0$ and $\phi(T) = r_0^2 T/8$, where $r_0 > 0$ is a constant. The amplitude can be any positive real number and the frequency is $\omega = 1 + \epsilon\phi'(T) + O(\epsilon^2) = 1 + \epsilon r_0^2/8 + O(\epsilon^2)$.

Especially, when $x(0) = a$ and $\dot{x}(0) = 0$, we have

$$r_0\cos\left(\tau + \frac{r_0^2 T}{8}\right)\Bigg|_{\tau=0,T=0} = r_0 = a, -r_0\sin\left(\tau + \frac{r_0^2 T}{8}\right)\Bigg|_{\tau=0,T=0} = 0$$

Therefore, the solution is

$$x(t,\epsilon) = a\cos\left(\left(1+\frac{a^2\epsilon}{8}\right)t\right) + O(\epsilon)$$

7.6.9

Similar to 7.6.5, the expression of h is

$$h\left(x_0, \frac{\partial x_0}{\partial \tau}\right) = h(r\cos\theta, -r\sin\theta) = -r^5\sin^3\theta\cos^2\theta + r^3\sin^3\theta$$

The equation for r is

$$\dot{r} = \langle h\sin\theta\rangle = -r^5\langle\sin^4\theta\cos^2\theta\rangle + r^3\langle\sin^4\theta\rangle = -\frac{r^5}{16} + \frac{3r^3}{8}$$

and the equation for $\dot{\phi}$ is

$$r\dot{\phi} = \langle h\cos\theta\rangle = -r^5\langle\sin^3\theta\cos^3\theta\rangle + r^3\langle\sin^3\theta\cos\theta\rangle = 0$$

Therefore, by obtaining the solution of $\dot{r} = 0$, we have the amplitude is $r = \sqrt{6}$. The frequency is $\omega = 1 + O(\epsilon^2)$ since $\phi(T) = 0$.

The solution of x is then

$$x(t,\epsilon) = \sqrt{6}\cos t + O(\epsilon)$$

8 Bifurcations Revisited

8.1 Saddle-Node, Transcritical and Pitchfork Bifurcations

8.1.6

The fix points are

$$x^* = 1 \pm \sqrt{1-\mu}, y^* = 2 \pm 2\sqrt{1-\mu}$$

Clearly, the critical point is $\mu_c = 1$. When $\mu > \mu_c$, there is no fix point; when $\mu < \mu_c$, there are two fix points; when $\mu = \mu_c$, there are two identical fix points. Therefore, it is a saddle-node bifurcation.

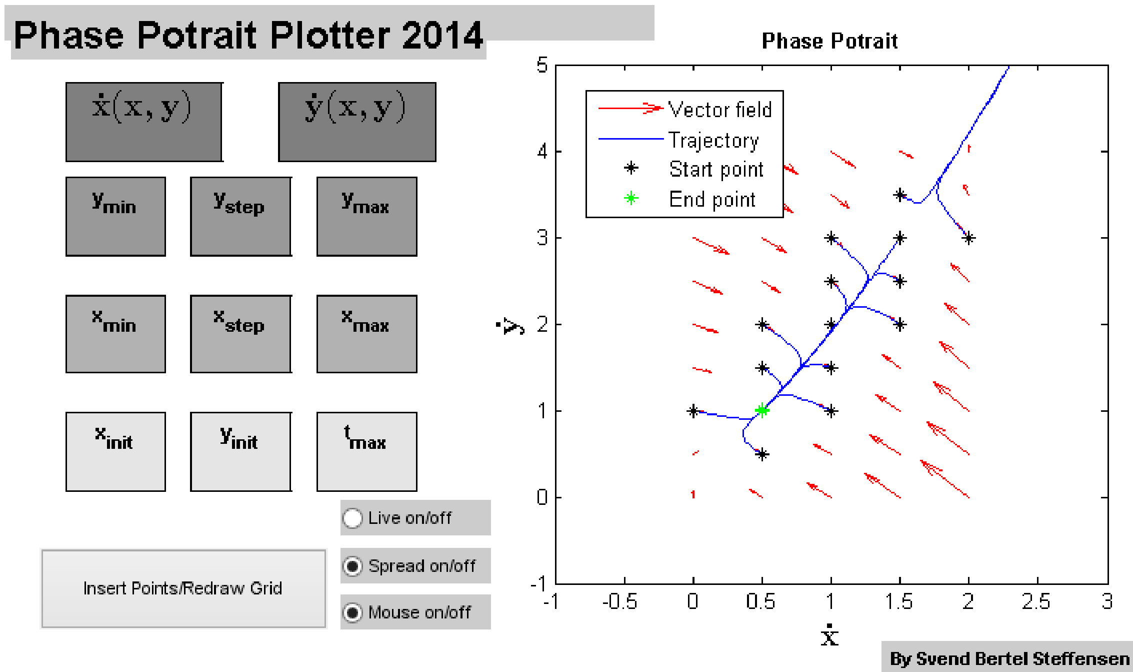

Figure 2: 8.1.6. $\mu = 3/4$

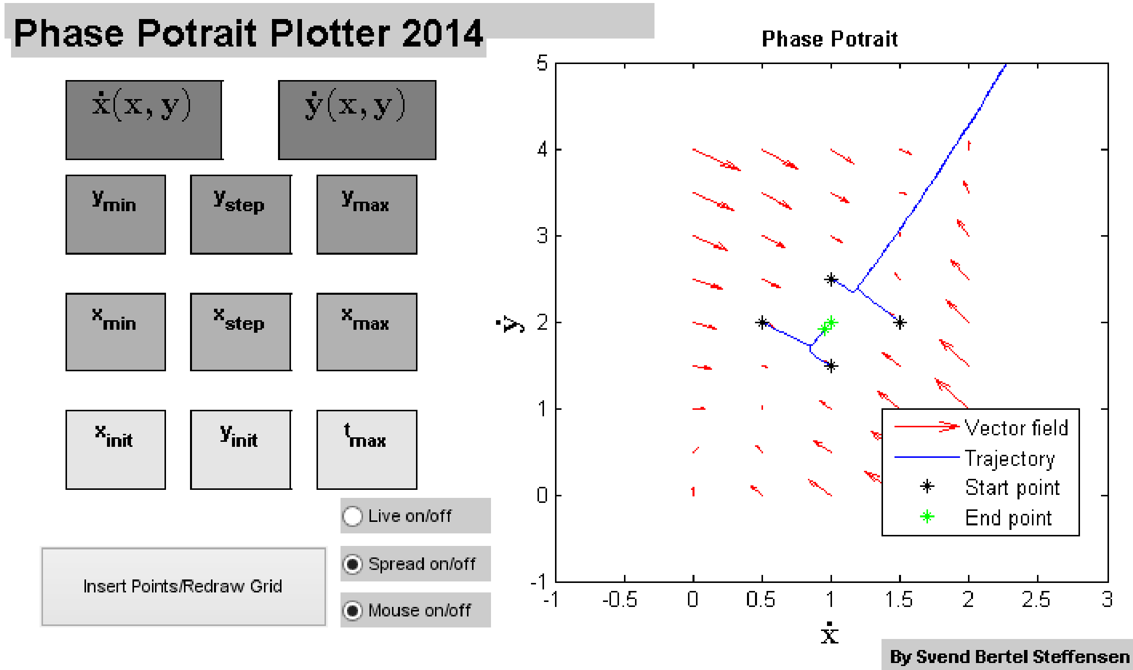

Figure 3: 8.1.6. $\mu = 1$

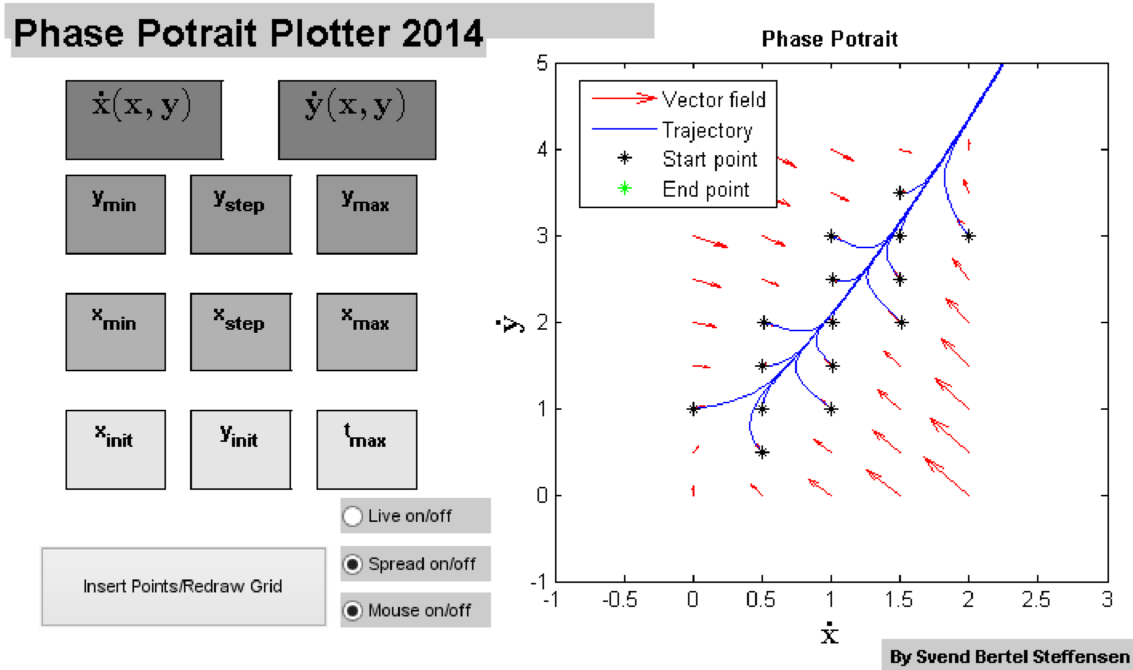

Figure 4: 8.1.6. $\mu = 5/4$