Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 07

February 23, 2015

6 Phase Plane

6.8 Index Theory

6.8.2

The fix point is $(0, 0)$ and $I = 0$ since the velocity arrow always points to $+x$ direction.

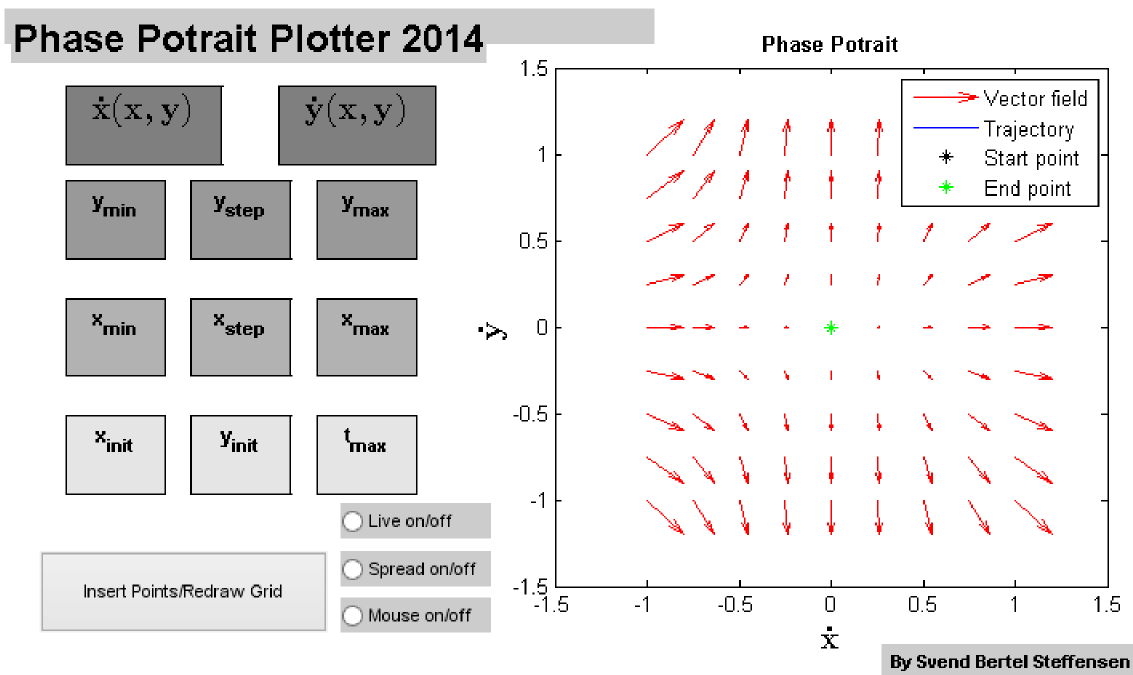


Figure 1: 6.8.2

6.8.4

The fix point is $(0, 0)$ and $I = -1$ since a negative x results in a negative velocity in y.

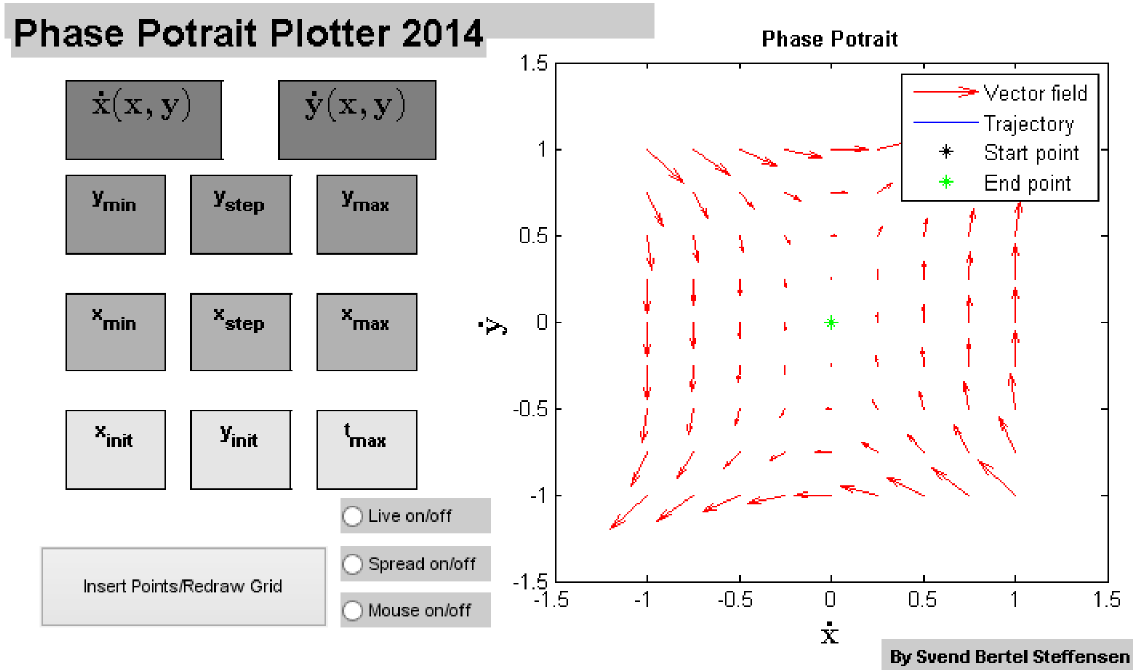


Figure 2: 6.8.4

7 Limit Cycles

7.2 Ruling Out Closed Orbits

7.2.6

(a) First integrate f gives

$$V(x,y) = -\int f(x,y)\mathrm{d}x = -\int (y^2 + y\cos x)\mathrm{d}x = -xy^2 - y\sin x + h(y)$$

where h is only a function of y. Then solving $g(x,y) = -\partial V(x,y)/\partial y$ gives

$$-\frac{\partial V(x,y)}{\partial y} = \frac{\partial(xy^2 + y\sin x - h(y))}{\partial y} = 2xy + \sin x - h'(y)$$

Therefore, $h'(y) = 0$ and $h(y) = C$ where C is a constant. The potential is then

$$V(x,y) = -xy^2 - y\sin x + C$$

(b) Integrate f gives

$$V(x,y) = -\int f(x,y)\mathrm{d}x = -\int (3x^2 - 1 - \mathrm{e}^{2y})\mathrm{d}x = -x^3 + x + x\mathrm{e}^{2y} + h(y)$$

Then we have

$$-\frac{\partial V(x,y)}{\partial y} = \frac{\partial(x^3 - x - x\mathrm{e}^{2y} - h(y))}{\partial y} = -2x\mathrm{e}^{2y} - h'(y)$$

Therefore $h'(y) = 0$, $h(y) = C$ and

$$V(x,y) = -x^3 + x + x\mathrm{e}^{2y} + C$$

7.2.10

The only real fix point is $(x^*, y^*) = (0,0)$. For the Lyapunov function V, we should have

$$V(x^*, y^*) = 0, V(x,y) = ax^2 + by^2 > 0 \forall x \neq x^*, y \neq y^*$$

This gives that $a, b > 0$. Then the derivative of V is

$$\dot{V} = 2ax\dot{x} + 2by\dot{y} = 2axy - 2bxy - 2ax^4 - 2by^4$$

Obviously $\dot{V}(x^*, y^*) = 0$. A necessary condition for $\dot{V}(x,y) < 0, \forall x \neq x^*, y \neq y^*$ is $a = b$. Therefore, by choosing values of a and b such that $a = b > 0$, $V = ax^2 + by^2$ is a valid Lyapunov function, and there is no closed orbit around the origin.

7.3 Poincare-Bendixson Theorem

7.3.4

(a) The Jacobian is

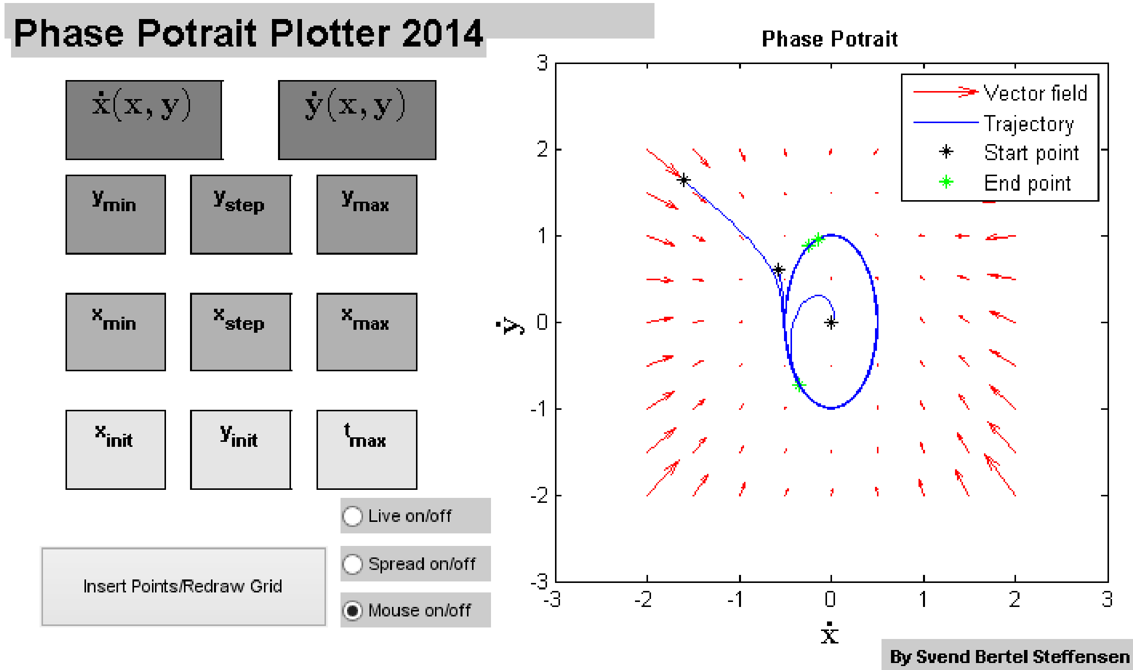
$$\mathbf{J}|_{(0,0)} = \begin{bmatrix} 1 - 12x^2 - y^2 - \frac{1}{2}y & -2xy - \frac{1}{2} - \frac{1}{2}x \\ -8xy + 2 + 4x & 1 - 4x^2 - 3y^2 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

and the eigenvalues are $\lambda_{1,2} = 1 \pm i$. This indicates that it is a unstable spiral.

(b) Note that $V = 0$ gives $4x^2 + y^2 = 1$. For all other (x,y), it can be shown that $V(x,y) > 0$. The derivative of V is

$$\dot{V} = 2(1 - 4x^2 - y^2)(-8x\dot{x} - 2y\dot{y}) = -4(1 - 4x^2 - y^2)^2(4x^2 + y^2)$$

Clearly, for (x,y) on the ellipse, $\dot{V} = 0$. For $(x,y) = (0,0)$, $\dot{V} = 0$ but this point is unstable. For all other (x,y), $\dot{V} < 0$. Therefore, V is a Lyapunov-like function and the all the trajectories converge to the ellipse.


Phase Potrait Plotter 2014
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Live on/off
Insert Points/Redraw Grid
Spread on/off
Mouse on/off
Phase Potrait
Vector field
Trajectory
Start point
End point
3
2
1
0
-1
-2
-3
ẏ
ẋ
By Svend Bertel Steffensen

Figure 3: 7.3.4