Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 06

February 23, 2015

6 Phase Plane

6.3 Fixed Points and Linearization

6.3.10

(a) Linearizing the equation around the origin and the Jacobian is

$$\mathbf{J}|_{(0,0)} = \begin{bmatrix} y & x \\ 2x & -1 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues are $\lambda_{1,2} = 0, -1$ and the fact that $\lambda_1 = 0$ indicates that it is a non-isolated fixed point.

(b) Solving $\dot{x} = 0, \dot{y} = 0$ only yields one solution. So the origin is an isolated fix point.

(c) It's a saddle node. The origin is stable along $\mathbf{v}_2$ and unstable along $\mathbf{v}_1$.

(d)

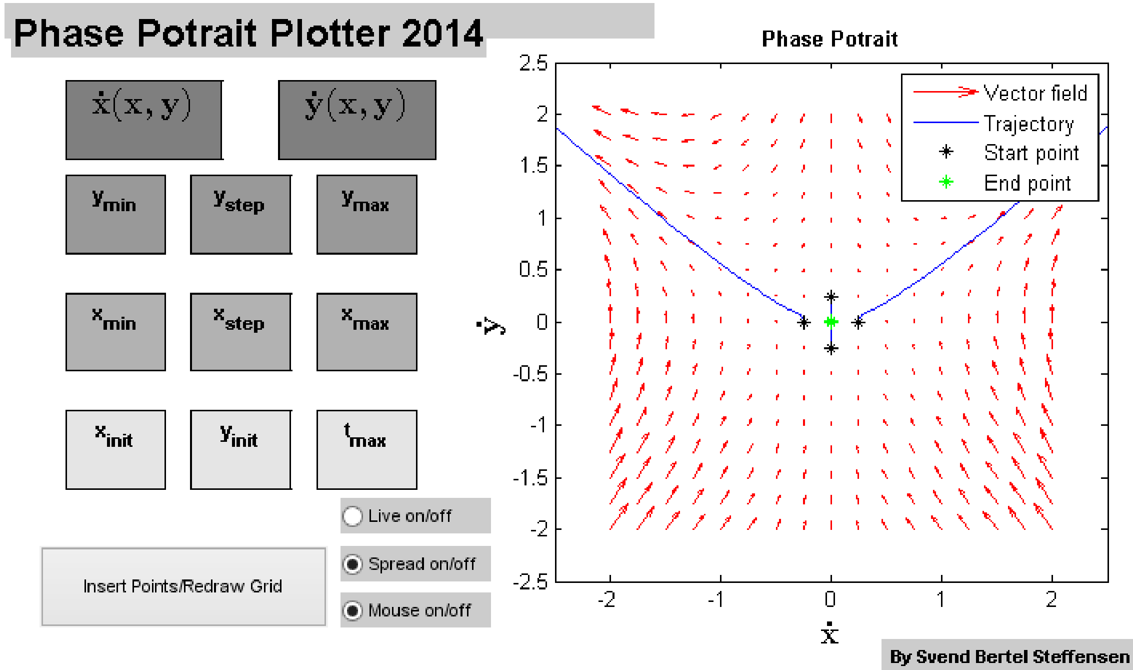

Figure 1: 6.3.10

6.5 Conservative Systems

6.5.1

Rewrite the system as

$$\dot{x} = y, \dot{y} = x^3 - x$$

(a) The equilibrium points are $(-1,0)$, $(0,0)$ and $(1,0)$.

- For $(0,0)$, the Jacobian is

$$\mathbf{J}|_{(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ 3x^2-1 & 0 \end{array}\right]\Bigg|_{(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right]$$

  and the eigenvalues are $\lambda_{1,2} = \pm i$. So it is a center.

- For $(\pm1,0)$, the Jacobian is

$$\mathbf{J}|_{(\pm1,0)} = \left[\begin{array}{cc} 0 & 1 \\ 2 & 0 \end{array}\right]$$

  The eigenvalues are $\lambda_{1,2} = \pm\sqrt{2}$, so it is saddle node.

(b) The kinetic energy is $\dot{x}^2/2$. The potential is

$$V = -\int (x^3 - x)\mathrm{d}x = \frac{1}{2}x^2 - \frac{1}{4}x^4 + C_0$$

where C_0 is a constant. Therefore, the conservation law is

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 + C_0$$

(c)

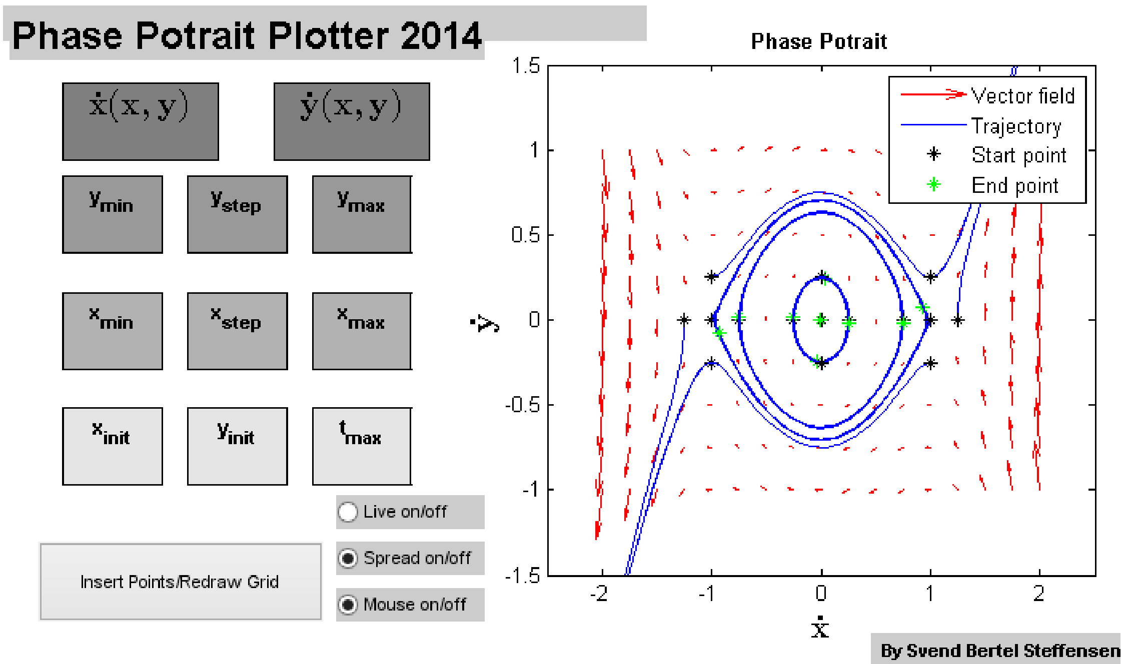

Figure 2: 6.5.1

6.5.11

The fix points are $(0,0)$,$(1,0)$ and $(-1,0)$.

- For $(\pm 1, 0)$, the Jacobian is

$$\mathbf{J}|_{(\pm1,0)} = \left[\begin{array}{cc} 0 & 1 \\ 1-3x^2 & -b \end{array}\right]\Bigg|_{(\pm1,0)} = \left[\begin{array}{cc} 0 & 1 \\ -2 & -b \end{array}\right]$$

  and the eigenvalues are $\lambda_{1,2} = (-b \pm \sqrt{b^2-8})/2$. It is a stable spiral.

- For $(0,0)$, the Jacobian is

$$\mathbf{J}|_{(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ 1 & -b \end{array}\right]$$

  The eigenvalues are $\lambda_{1,2} = (-b \pm \sqrt{4+b^2})/2$. It is a saddle node.

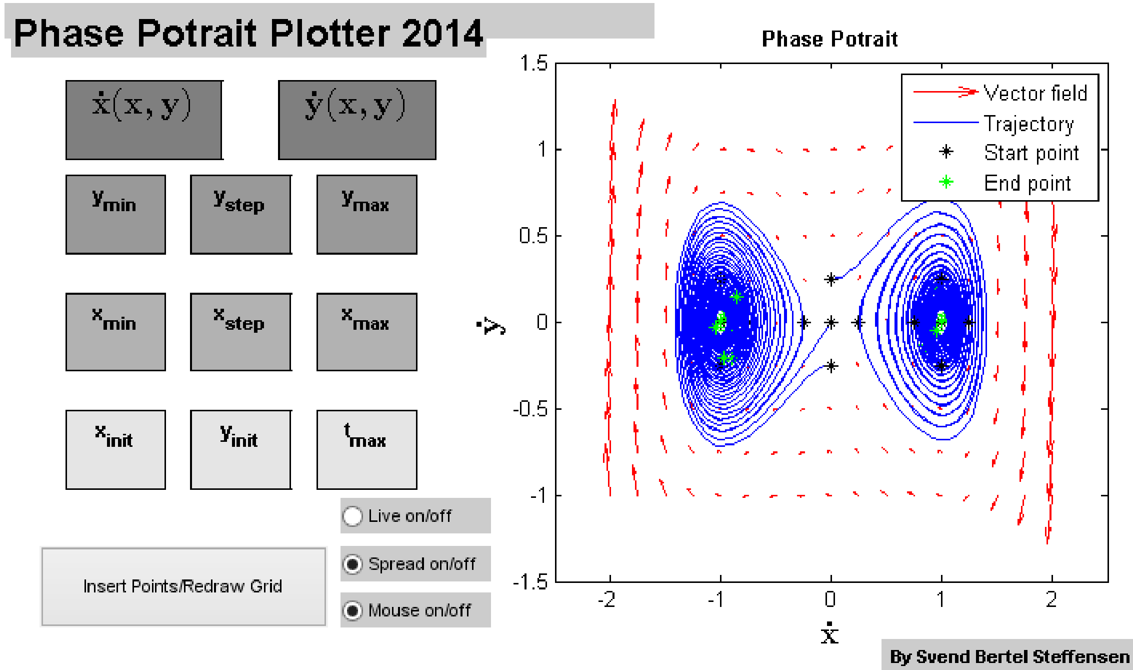

Figure 3: 6.5.11

6.5.13

(a) First we need to show that it is a conservative system. The kinetic energy is $\dot{x}^2/2$ and the potential is

$$V(x) = \int (x + \epsilon x^3)\mathrm{d}x = C + \frac{1}{2}x^2 + \frac{1}{4}\epsilon x^4$$

So the energy is

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}\epsilon x^4 + C$$

Especially,

$$\dot{E} = \dot{x}\ddot{x} + x\dot{x} + \epsilon x^3\dot{x} = \dot{x}\left(\ddot{x} + x + \epsilon x^3\right) = 0$$

So the system is conservative.

We then need to show that $x = 0$ is the minimum of the total energy E. Note that when $\epsilon > 0$, we have $E \geq C$ since the square of a real number is non-negative. At the origin, we have $x = 0, \dot{x} = 0$ and $E(0) = C$. Therefore, it is a minimum of E, so the origin is a nonlinear center.

(b) When $\epsilon < 0$, for the trajectories closed to the origin, we have $x^2 \gg x^4$. Therefore, E is still an increasing function of x and $\dot{x}$, so the trajectories are closed. When the trajectories are far away, $x^2 \gg x^4$ no longer holds, so they may not be closed.

6.6 Reversible Systems

6.6.7

Rewrite the 2nd order ODE as

$$\dot{x} = y, \dot{y} = -xy - x$$

Denote the transitions as $x \to -\tilde{x}$, $y \to \tilde{y}$ and $t \to -\tilde{t}$, and we have

$$\dot{x} = y \to \frac{\mathrm{d}x}{\mathrm{d}t} = y \to \frac{-\mathrm{d}\tilde{x}}{-\mathrm{d}\tilde{t}} = \tilde{y} \to \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} = \tilde{y}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -xy - x \to \frac{\mathrm{d}\tilde{y}}{-\mathrm{d}\tilde{t}} = \tilde{x}\tilde{y} + \tilde{x} \to \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}} = -\tilde{x}\tilde{y} - \tilde{x}$$

Therefore, the system is reversible.

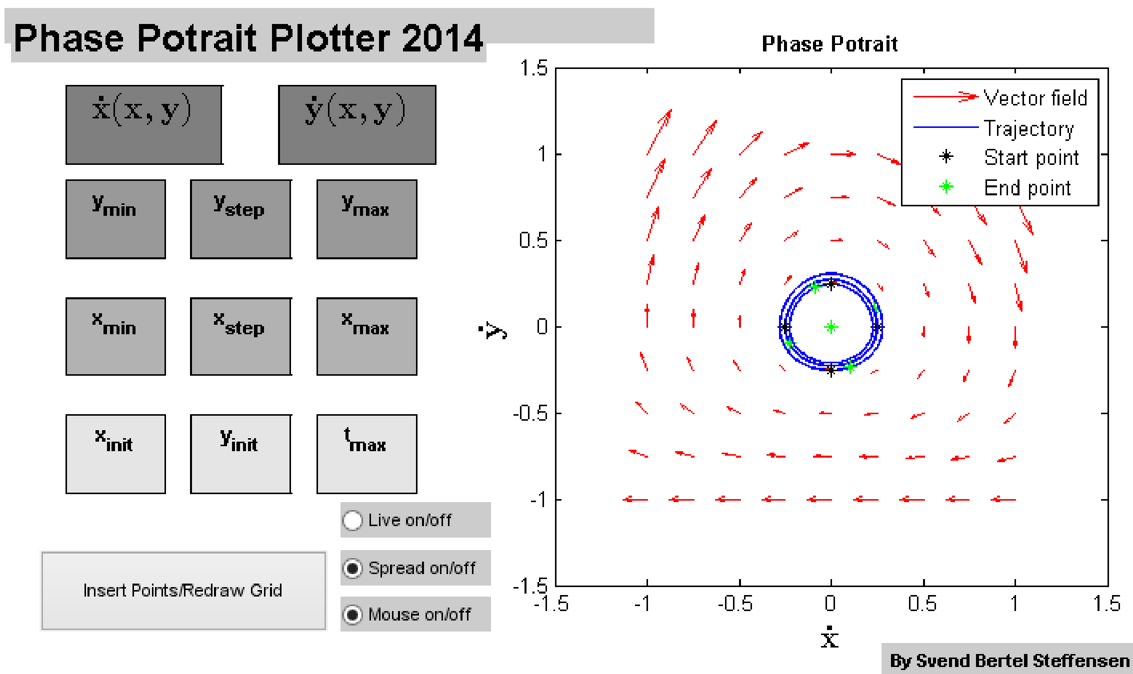

Figure 4: 6.6.7