Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 05

February 16, 2015

6 Phase Plane

6.1 Phase Protraits

6.1.2

Fix points: $(-1,0)$, $(0,0)$ and $(1,0)$. All of the 3 points are stable along y axis, but only $(-1,0)$ and $(1,0)$ are stable along x axis. Therefore, $(-1,0)$ and $(1,0)$ are stable and $(0,0)$ is saddle.

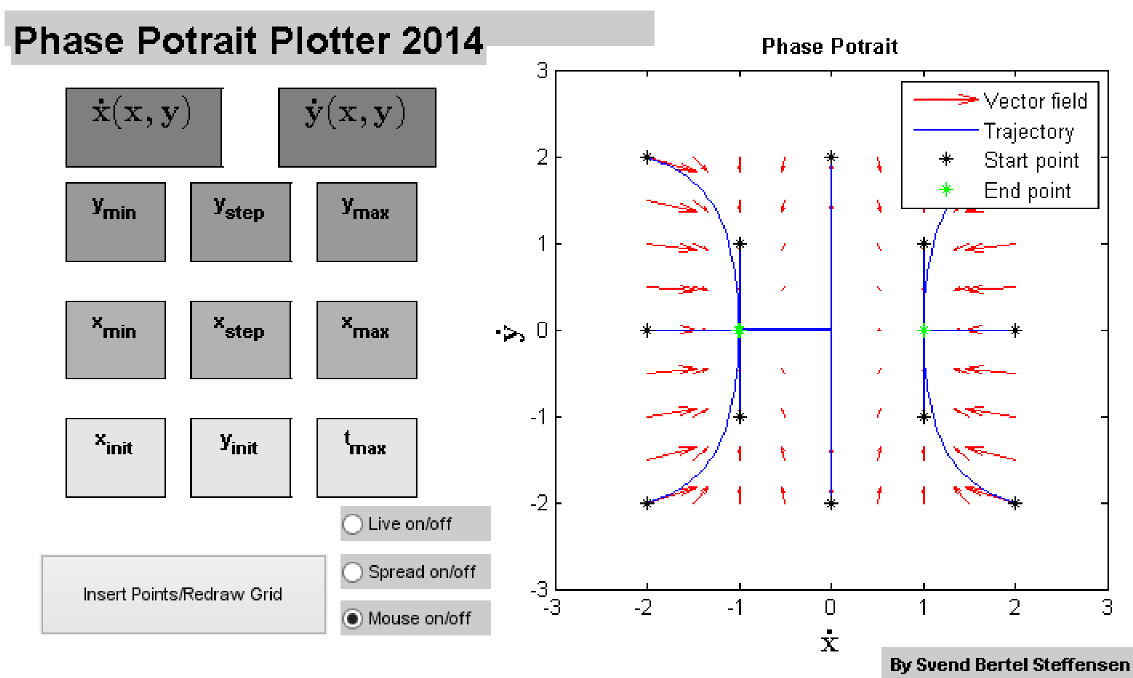

Figure 1: 6.1.2. Note that initial points $(0, \pm 1)$ show wrong results, due to numerical issues.

6.1.5

The fix point is $(0,0)$ and $(1,1)$. First linearize the system around the fix point $(0,0)$ and the Jacobian is

$$\mathbf{J}|_{(x,y)=(0,0)} = \begin{bmatrix} 2-2x-y & -x \\ 1 & -1 \end{bmatrix}\Bigg|_{(x,y)=(0,0)} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues of the Jacobian are $\lambda_1 = 2$ and $\lambda_2 = -1$ and the corresponding eigenvector is $\mathbf{v}_1 = [1, 1/3]'$ and $\mathbf{v}_2 = [0, 1]'$. Therefore it is unstable along the direction of $\mathbf{v}_1$ and is stable along the direction of $\mathbf{v}_2$. The fix point is a saddle node.

We then linearize the system around $(1,1)$ and the Jacobian is

$$\mathbf{J}|_{(x,y)=(1,1)} = \begin{bmatrix} 2-2x-y & -x \\ 1 & -1 \end{bmatrix}\Bigg|_{(x,y)=(1,1)} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

and the eigenvalues are $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$. Since $\Re\lambda_{1,2} < 0$ and $\Im\lambda_{1,2} \neq 0$, it is a stable spiral.

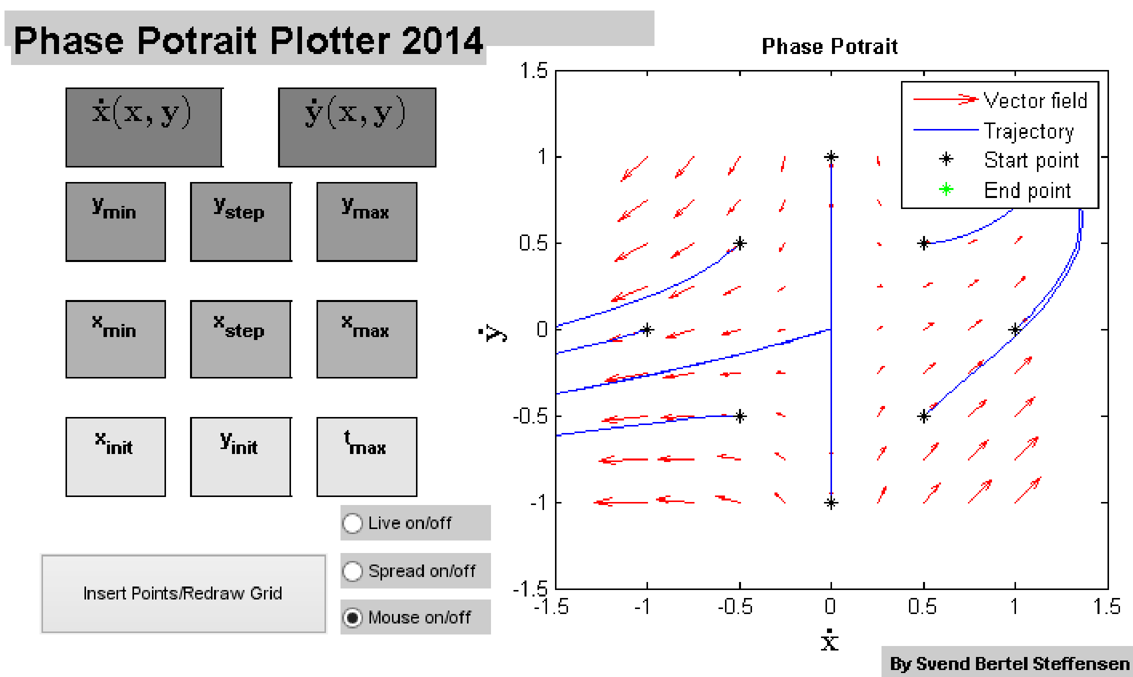

Figure 2: 6.1.5. Note that initial points $(0, \pm 1)$ show wrong results, due to numerical issues.

6.3 Fixed Points and Linearization

6.3.1

The fix points are $(2,2)$ and $(-2,-2)$.

- For $(2,2)$, the Jacobian is

$$\mathbf{J}|_{(2,2)} = \left[\begin{array}{cc} 1 & -1 \\ 2x & 0 \end{array} \right] \Bigg|_{(2,2)} = \left[\begin{array}{cc} 1 & -1 \\ 4 & 0 \end{array} \right]$$

The eigenvalues are $\lambda_1 = (1+\sqrt{15}i)/2$, $\lambda_2 = (1-\sqrt{15}i)/2$ and the corresponding eigenvectors are $\mathbf{v}_1 = [1,(1-\sqrt{15}i)/2]'$, $\mathbf{v}_2 = [1,(1+\sqrt{15}i)/2]'$. The fix point is unstable spiral.

- For $(-2,-2)$, the Jacobian is

$$\mathbf{J}|_{(-2,-2)} = \left[\begin{array}{cc} 1 & -1 \\ -4 & 0 \end{array} \right]$$

The eigenvalues are $\lambda_1 = (1+\sqrt{17})/2$, $\lambda_2 = (1-\sqrt{17}/2$ and the corresponding eigenvectors are $\mathbf{v}_1 = [1,(1-\sqrt{17})/2]'$, $\mathbf{v}_1 = [1,(1+\sqrt{17})/2]'$. The fix point is unstable along $\mathbf{v}_1$ and stable along $\mathbf{v}_2$. It is a saddle node.

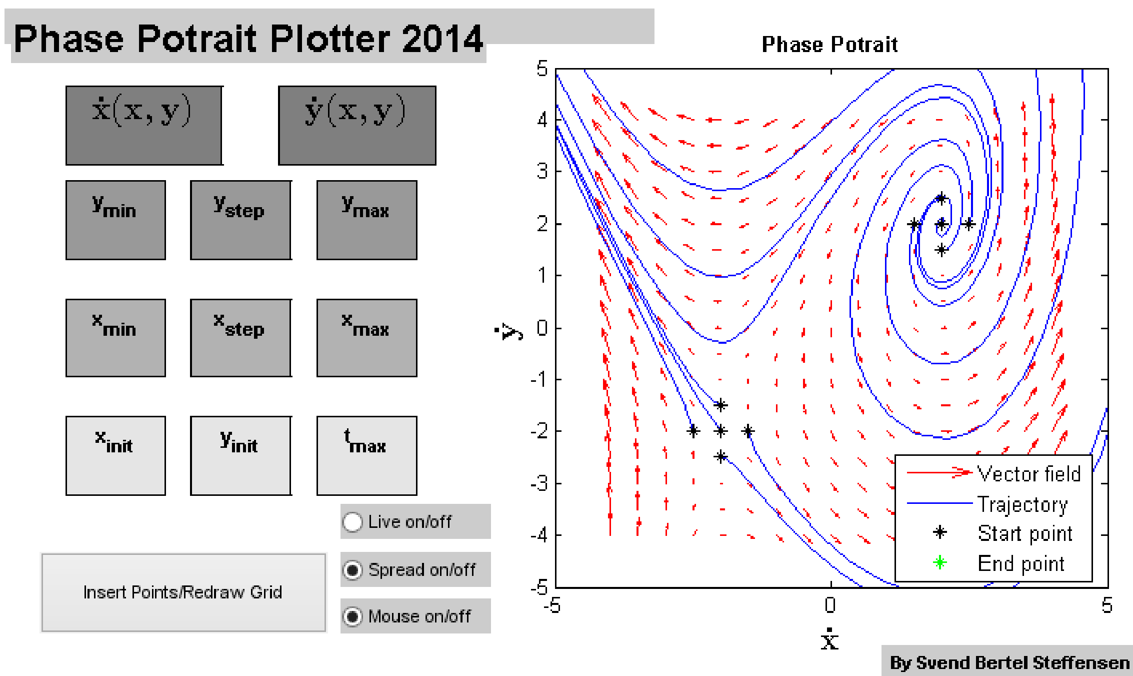

Figure 3: 6.3.1. Note that initial points $(2,2), (-2,-2)$ show wrong results, due to numerical issues.

6.3.4

The fix points are $(1,0)$, $(0,0)$ and $(-1,0)$

- For $(0,0)$, the Jacobian is

$$\mathbf{J}|_{(0,0)} = \left[\begin{array}{cc} 1-3x^2 & 1 \\ 0 & -1 \end{array} \right] \Bigg|_{(0,0)} = \left[\begin{array}{cc} 1 & 1 \\ 0 & -1 \end{array} \right]$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$ and the eigenvectors are $\mathbf{v}_1 = [1, 0]'$ and $\mathbf{v}_2 = [-1/2, 1]'$. The fix point is stable along $\mathbf{v}_1$ and unstable along $\mathbf{v}_2$. It is a saddle node.

- For $(\pm 1, 0)$, the Jacobian is

$$\mathbf{J}|_{(0,0)} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -1$ and the eigenvectors are $\mathbf{v}_1 = [1, 0]'$, $\mathbf{v}_2 = [1, 1]'$. The two fix points are stable nodes.

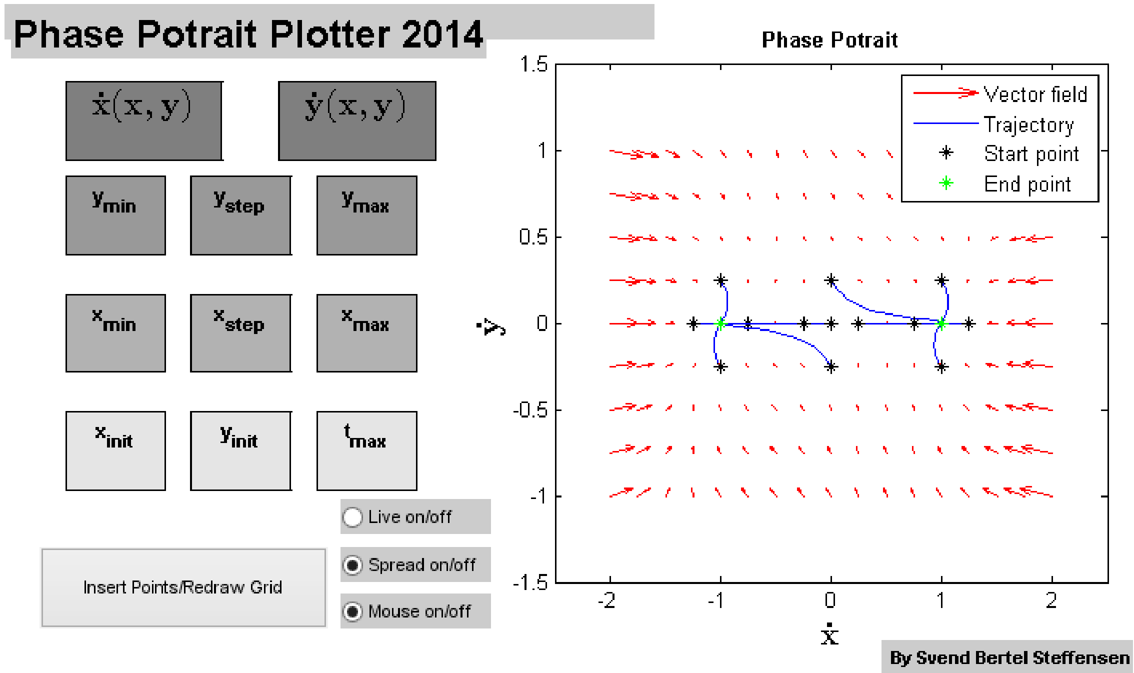

Figure 4: 6.3.4. Note that the initial point $(0, 0)$ shows wrong results, due to numerical issues.

6.3.5

The fix points are $(k_x\pi + \pi/2, k_y\pi), k_x, k_y \in \mathbb{Z}$.

- For $(2m\pi + \pi/2, 2n\pi), m, n \in \mathbb{Z}$, the Jacobian is

$$\mathbf{J}|_{(2m\pi+\pi/2,2n\pi)} = \begin{bmatrix} 0 & \cos y \\ -\sin x & 0 \end{bmatrix}\Bigg|_{(2m\pi+\pi/2,2n\pi)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues are $\lambda_{1,2} = \pm i$. It is a center.

- For $(2m\pi + \pi/2, 2n\pi + \pi), m, n \in \mathbb{Z}$, the Jacobian is

$$\mathbf{J}|_{(2m\pi+\pi/2,2n\pi+\pi)} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$ and the eigenvectors are $\mathbf{v}_1 = [1, -1]'$, $\mathbf{v}_2 = [1, 1]'$. The fix point is unstable along $\mathbf{v}_1$ and stable along $\mathbf{v}_2$. It is a saddle node.

- For $(2m\pi - \pi/2, 2n\pi), m, n \in \mathbb{Z}$, the Jacobian is

$$\mathbf{J}|_{(2m\pi-\pi/2,2n\pi)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$ and the eigenvectors are $\mathbf{v}_1 = [1, 1]'$, $\mathbf{v}_2 = [1, -1]'$. The fix point is unstable along $\mathbf{v}_1$ and stable along $\mathbf{v}_2$. It is a saddle node.

- For $(2m\pi - \pi/2, 2n\pi + \pi), m, n \in \mathbb{Z}$, the Jacobian is

$$\mathbf{J}|_{(2m\pi-\pi/2,2n\pi+\pi)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues are $\lambda_{1,2} = \pm i$, and it is a center.

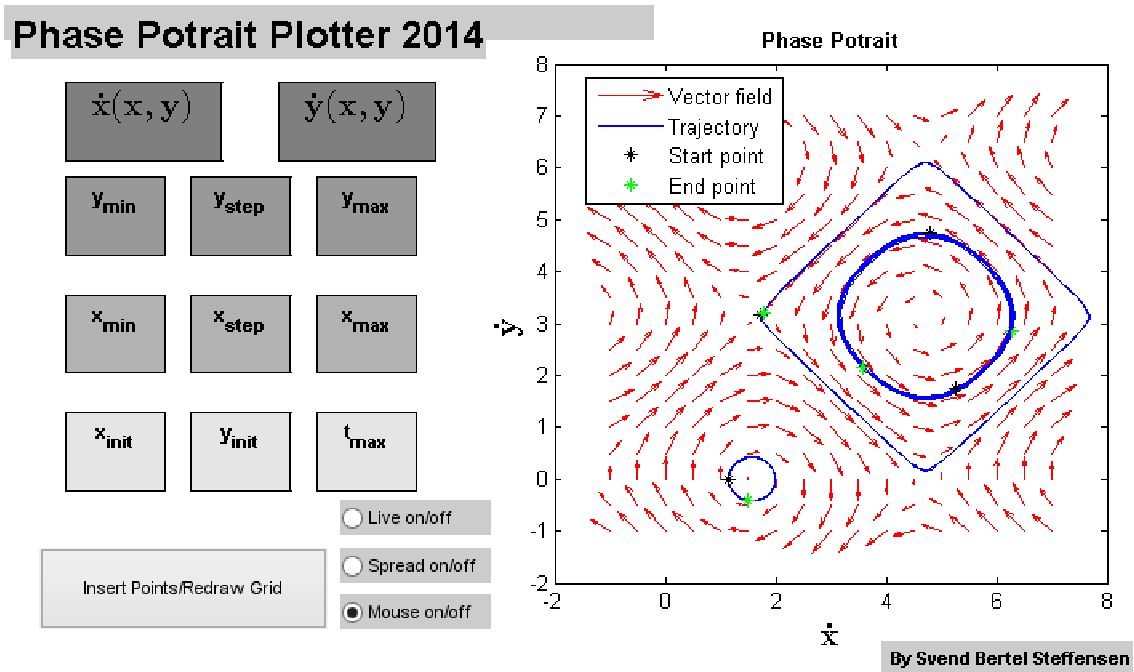

Figure 5: 6.3.5

6.3.8

(a) Denote the mass of the particle as m. The direction of $\mathbf{x}$ is from m_1 to m_2. The Newton's Second Law gives that

$$\mathbf{F} = -G\frac{m_1 m}{x^2} + G\frac{m_2 m}{(a-x)^2}$$

Since $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{x}}$, we then have

$$\ddot{\mathbf{x}} = -\frac{Gm_1}{x^2} + \frac{Gm_2}{(x-a)^2}$$

(b) Rewrite the second-order ODE as

$$\dot{x} = y, \dot{y} = -\frac{Gm_1}{x^2} + \frac{Gm_2}{(x-a)^2}$$

and the fix point is $(a/(\sqrt{m_2/m_1}+1), 0)$. The Jacobian is

$$\mathbf{J}|_{(a/(\sqrt{m_2/m_1}+1),0)} = \left[\begin{array}{cc} 0 & 1 \\ 2\frac{Gm_1}{x^3} - 2\frac{Gm_2}{(x-a)^3} & 0 \end{array} \right]\Bigg|_{(a/(\sqrt{m_2/m_1}+1),0)} = \left[\begin{array}{cc} 0 & 1 \\ \frac{2Gm_1}{a^3}\left(\sqrt{\frac{m_2}{m_1}}+1\right)^3\left(\sqrt{\frac{m_1}{m_2}}+1\right) & 0 \end{array} \right]$$

The eigenvalues are $\lambda_{1,2} = \pm\sqrt{\frac{2Gm_1}{a^3}\left(\sqrt{\frac{m_2}{m_1}}+1\right)^3\left(\sqrt{\frac{m_1}{m_2}}+1\right)}$. So it is unstable.

6.3.14

A linearization predicts that

$$\mathbf{J}_{(0,0)} = \left[\begin{array}{cc} 3ax^2 & -1 \\ 1 & 3ay^2 \end{array} \right]\Bigg|_{(0,0)} = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]$$

The eigenvalues are $\lambda_{1,2} = \pm i$, so it is a center.

But it is worth noticing that according to Example 6.3.2, linearization doesn't work for this problem. Instead, by transforming to polar coordinate, we have $x = r\cos\theta$ and $y = r\sin\theta$. So we have

$$x\dot{x} + y\dot{y} = r\dot{r} \to \dot{r} = ar^3(\cos^4\theta + \sin^4\theta)$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x^2 + axy^3 + y^2 - ax^3y}{r^2} = 1 + ar^2\cos\theta\sin\theta(\sin^2\theta - \cos^2\theta)$$

So if $a = 0$, the origin is a center; if $a < 0$, the origin is stable; if $a > 0$, the origin is unstable. Note that for $a \neq 0$, $\dot{\theta} \neq 0$, so it's spiral.

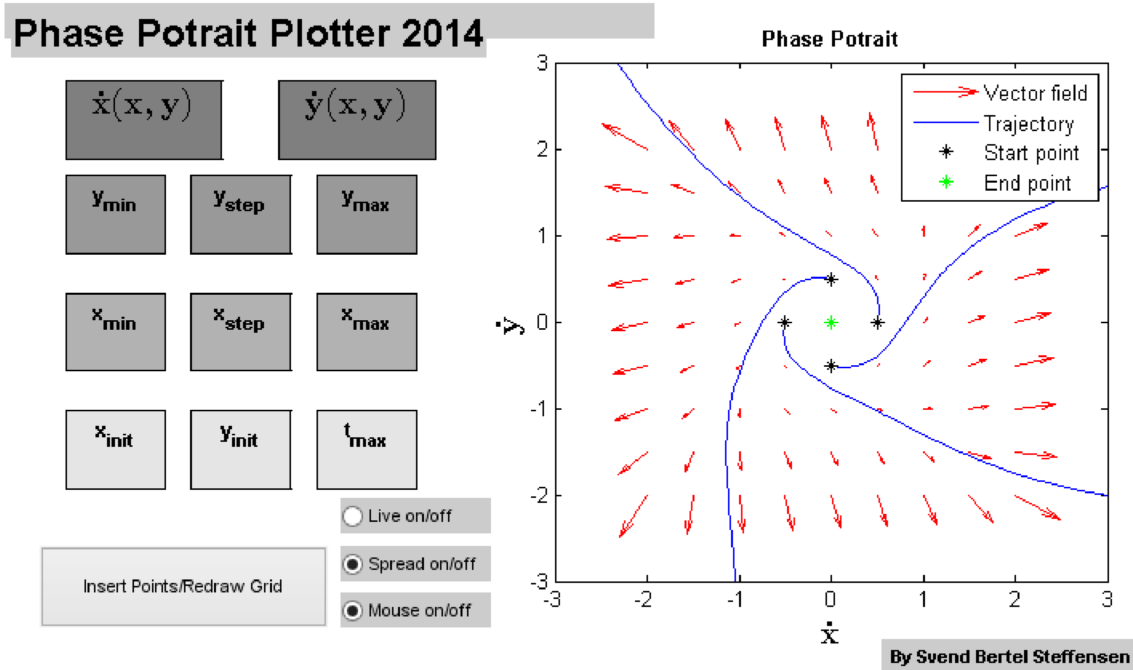

Figure 6: 6.3.14, $a = 1$

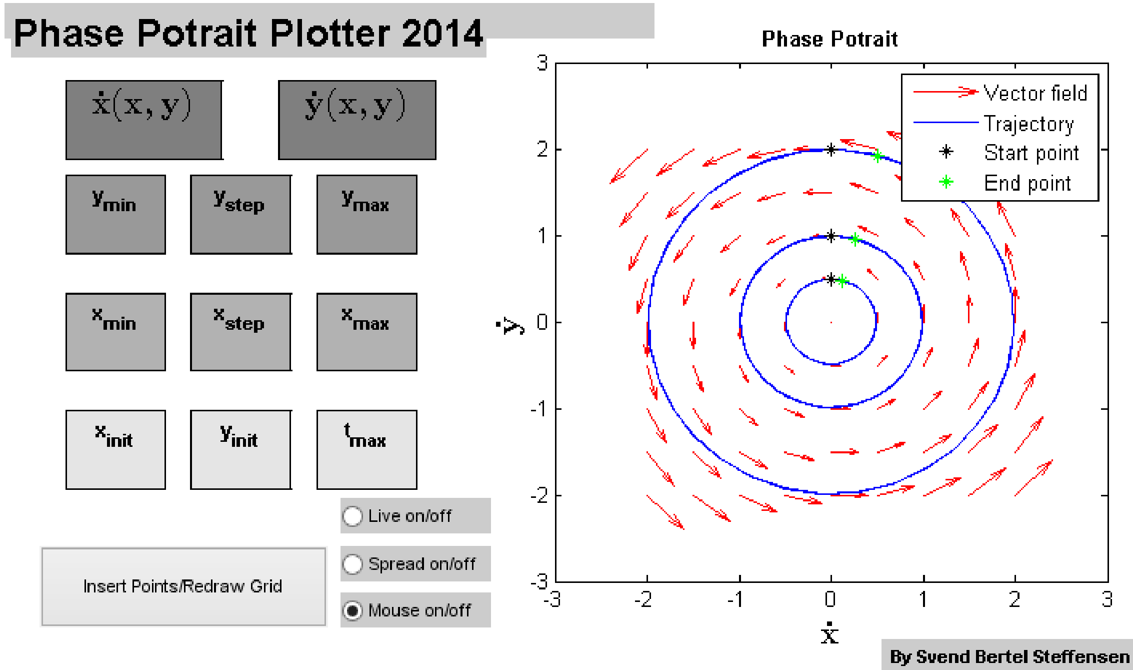

Figure 7: 6.3.14, $a = 0$

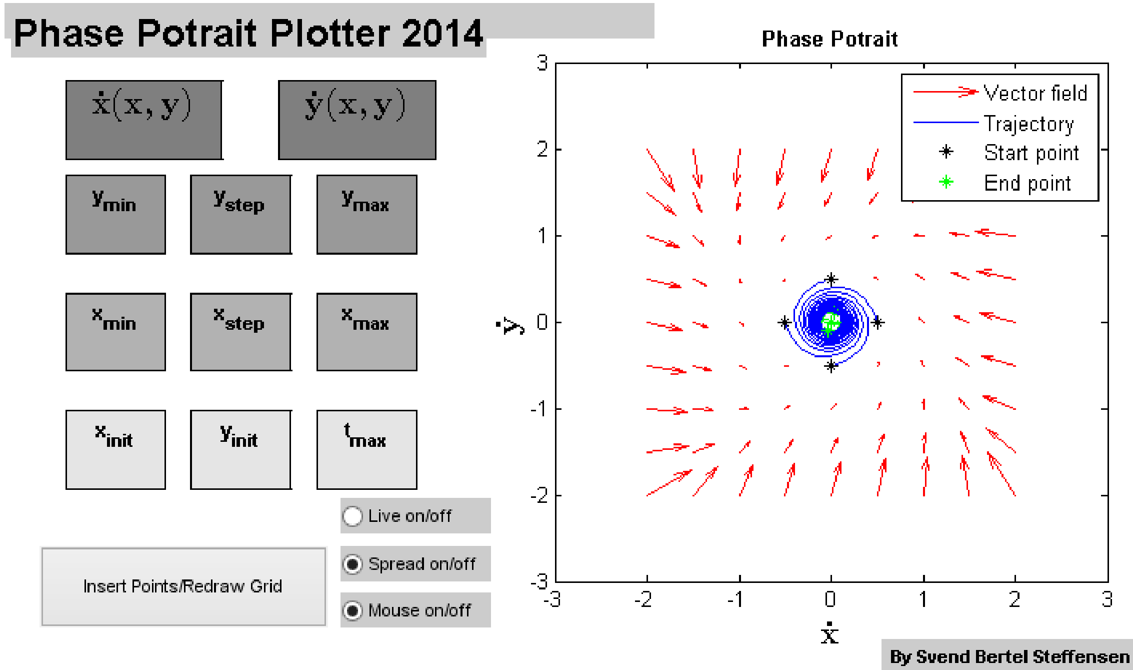

Figure 8: 6.3.14, $a = -1$

As a general note, the linearization doesn't work when the prediction is at the border line, for example $\lambda = 0$ in 6.3.10 and center ($\Re\lambda = 0$) in this problem. In fact, there exists a theorem that a point is (exponentially) stable if and only if the real part of all the eigenvalues are negative; is unstable if the real part of some eigenvalues are positive. The linearization fails when the real part of all the eigenvalues are non-positive, but some are 0.

A general discussion is available in the following page.

`math.stackexchange.com/questions/337459/isolated-versus-non-isolated-fixed-point-2d-dynamics`