Dynamical Systems and Chaos 2015 Spring

Homework Solutions

February 16, 2015

4 Flows on the Circle

4.1 Examples and Definitions

4.1.1

A well-defined vector field on a circle requires $V(\theta) = V(\theta + 2k\pi), \forall k \in \mathbb{Z}$. Therefore,

$$\sin(a\theta) = \sin(a(\theta + 2k\pi)) = \sin(a\theta)\cos(2ak\pi) + \cos(a\theta)\sin(2ak\pi)$$

If $\cos(a\theta) = 0$, $\cos(2ak\pi) = 1$. Otherwise, divide both sides by $\cos(a\theta)$ and we have

$$\tan(a\theta)(1 - \cos(2ak\pi)) = \sin(2ak\pi)$$

Since $\tan(a\theta)$ can take any real value, we have $\cos(2ak\pi) = 1$ and $\sin(2ak\pi) = 0$. Therefore, $2ak\pi = 2m\pi, \forall k, m \in \mathbb{Z}$. This leads to $a \in \mathbb{Z}$, i.e. a should be an integer.

4.1.2

Assume $\theta \in [0, 2\pi)$. Solving $f(\theta^*) = 0$ leads to $\theta^* = 2\pi/3, 4\pi/3$.

- For $\theta^* = 2\pi/3$, $f'(\theta^*) = -2\sin\theta|_{\theta^*=2\pi/3} = -\sqrt{3}$. So it is a stable fix point.
- For $\theta^* = 4\pi/3$, $f'(\theta^*) = \sqrt{3}$. So it is an unstable fix point.

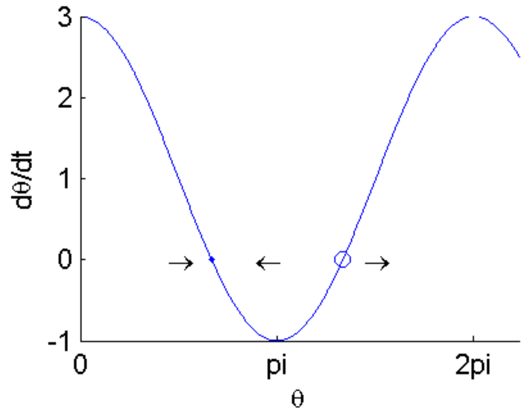

Figure 1: 4.1.2

4.1.8

(a) The potential is defined as

$$V(\theta) = -\int f(\theta)\mathrm{d}\theta = C - \sin\theta$$

where C is a constant. Since θ and $\theta + 2k\pi$ coincide for all integers k, we then should have $V(\theta) = V(\theta + 2k\pi)$. It is easy to show that this relation is satisfied.

(b) The potential is $V(\theta) = \theta + C$ where C is a constant. Clearly, $V(2\pi) \neq V(0)$, and therefore there exists no single-valued potential function.

(c) As indicated previously, $V(\theta) = V(\theta + 2k\pi), \forall\theta \in [0, 2\pi), k \in \mathbb{Z}$.

4.3 Nonuniform Oscillator

4.3.1

$$\begin{aligned}
T_{bottleneck} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{r + x^2} \\
&= \int_{-\pi/2}^{\pi/2} \frac{\sqrt{r}\mathrm{d}\theta}{\cos^2\theta} \frac{1}{r(1 + \tan^2\theta)} \\
&= \frac{1}{\sqrt{r}} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\theta}{\cos^2\theta} \cos^2\theta \\
&= \frac{1}{\sqrt{r}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \\
&= \frac{\pi}{\sqrt{r}}
\end{aligned}$$

5 Linear Systems

5.1 Definitions and Examples

5.1.1

(a) Divide $\dot{x}$ by $\dot{v}$ gives

$$\frac{\dot{x}}{\dot{v}} = \frac{v}{-\omega^2 x} \rightarrow \omega^2 x\dot{x} + v\dot{v} = 0$$

Integrate both sides and we have

$$\omega^2 \frac{x^2}{2} + \frac{v^2}{2} = C_0$$

where C_0 is a constant. Multiply both sides by 2 and we have

$$\omega^2 x^2 + v^2 = C$$

where $C = 2C_0$.

(b) $\frac{1}{2}m\omega^2 x^2$: potential; $\frac{1}{2}mv^2$: kinetic energy; $\frac{1}{2}mC$: total energy.

5.1.9

(a)

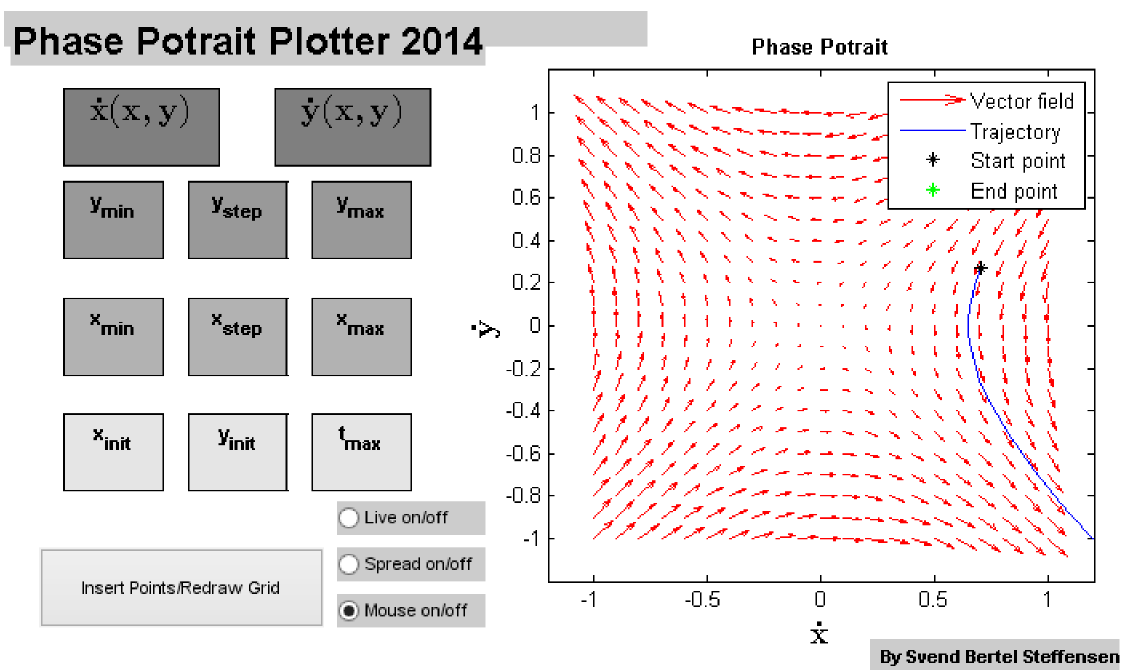

Figure 2: 5.1.9

(b) Since $x\dot{x} = y\dot{y} = -xy$, we have $x\dot{x} - y\dot{y} = 0$. Integrate the equation and we have

$$x\dot{x} - y\dot{y} = 0 \to \frac{1}{2}\frac{\mathrm{d}x^2}{\mathrm{d}t} - \frac{1}{2}\frac{\mathrm{d}y^2}{\mathrm{d}t} = 0 \to x^2 - y^2 = C$$

(c) The eigenvalues can be computed by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 0$$

The solutions are $\lambda_1 = 1$ and $\lambda_2 = -1$, and the corresponding eigenvectors are $[1, -1]'$ and $[1, 1]'$. Therefore the unstable manifold is $x + y = 0$ and the stable one is $x - y = 0$.

(d) From $u = x + y$, $v = x - y$, we have $x = (u + v)/2$ and $y = (u - v)/2$. Therefore

$$\dot{x} = -y, \dot{y} = -x \to \dot{u} + \dot{v} = v - u, \dot{u} - \dot{v} = -u - v \to \dot{u} = -u, \dot{v} = v$$

and the solutions are

$$u(t) = u_0\mathrm{e}^{-t}, v(t) = v_0\mathrm{e}^t$$

(e) Stable: $v = 0$, unstable: $u = 0$

(f)

$$\begin{aligned} x(t) =& \frac{u(t) + v(t)}{2} = \frac{u_0\mathrm{e}^{-t} + v_0\mathrm{e}^t}{2} = \frac{(x_0 + y_0)\mathrm{e}^{-t} + (x_0 - y_0)\mathrm{e}^t}{2} \\ y(t) =& \frac{u(t) - v(t)}{2} = \frac{u_0\mathrm{e}^{-t} - v_0\mathrm{e}^t}{2} = \frac{(x_0 + y_0)\mathrm{e}^{-t} - (x_0 - y_0)\mathrm{e}^t}{2} \end{aligned}$$

5.2 Classification of Linear Systems

5.2.2

(a) The eigenvalues can be computed by solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 2 = 0$$

and the solutions are $\lambda_{1,2} = 1 \pm i$. and the corresponding eigenvectors are $[i, 1]'$ and $[-i, 1]$.

(b) The solution is

$$\begin{aligned} \vec{x}(t) =& c_1\mathrm{e}^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + c_2\mathrm{e}^{(1-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix} \\ =& c_1\mathrm{e}^t(\cos t + i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix} + c_2\mathrm{e}^t(\cos t - i\sin t)\begin{bmatrix} -i \\ 1 \end{bmatrix} \\ =& \begin{bmatrix} -c_1\mathrm{e}^t\sin t - c_2\mathrm{e}^t\sin t + ic_1\mathrm{e}^t\cos t - ic_2\mathrm{e}^t\cos t \\ c_1\mathrm{e}^t\cos t + c_2\mathrm{e}^t\cos t + ic_1\mathrm{e}^t\sin t - ic_2\mathrm{e}^t\sin t \end{bmatrix} \\ =& (c_1 + c_2)\mathrm{e}^t\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i(c_1 - c_2)\mathrm{e}^t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \end{aligned}$$

5.2.4

Solving $5x + 10y = 0$ and $-x - y = 0$ gives the only fix point $(x, y) = (0, 0)$. To determine its stability, we have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 5-\lambda & 10 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 5 = 0$$

The solutions are $\lambda = 2 \pm i$. Since $\Re\lambda = 2 > 0$, the fix point is unstable; $\Im\lambda = \pm 1 \neq 0$, so it is spiral.

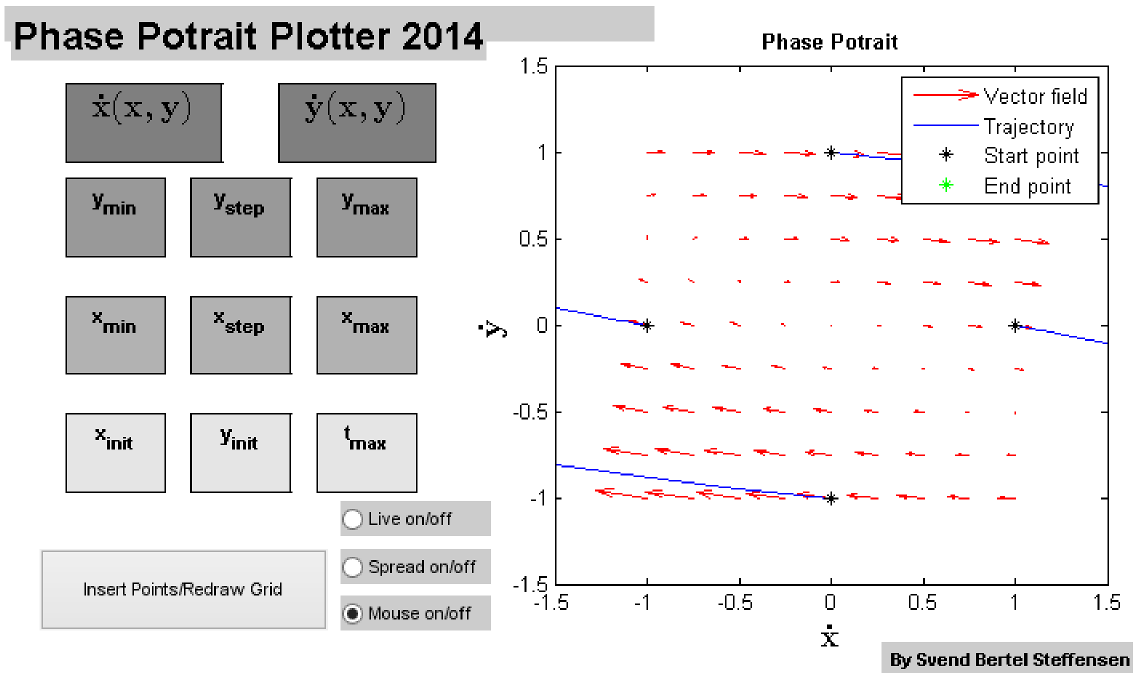

Figure 3: 5.2.4

Note, please change Line 296-299 of PhasePlot_2D_GUI_SBS.m from

```
syms t;
syms x(t) y(t)
V = odeToVectorField(diff(x) == Master_Struct.xdot, diff(y) == Master_Struct.ydot);
f = matlabFunction(V,'vars',{'t','Y'});
```

to

```
f1 = @(t,x,y) eval(Master_Struct.xdot);
f2 = @(t,x,y) eval(Master_Struct.ydot);
f = @(t,Y) [f1(t,Y(1),Y(2)); f2(t,Y(1),Y(2))];
```

because the original code may switch the coordinates. The new code is significantly slower, but it is correct.

5.2.11

The eigenvalues of the matrix is $\lambda_{1,2} = 0$, so $\mathbf{A}$ is non-semi-simple. There is a general theory (called "Jordan forms") to solve this type of initial values problems. However, here we use the naive method. First integrating $\dot{y}$ yields

$$y(t) = y_0 e^{\lambda t}$$

where y_0 is the initial value of y at time $t = 0$. Then plug in y into the equation of $\dot{x}$ and we have

$$\dot{x} = \lambda x + by = \lambda x + by_0 e^{\lambda t}$$

Again we use the variation of constant method. Solving $\dot{x} = \lambda x$ gives $x = Ce^{\lambda t}$. Assume C is a function and we have

$$\dot{C}e^{\lambda t} = by_0 e^{\lambda t}$$

Therefore $C = by_0 t + C_0$ where C_0 is a constant. We then determine the value of C_0 by having $x_0 = (by_0 t + C_0)e^{\lambda t}|_{t=0} = C_0$. Hence, the solution of the system is

$$x = (by_0 t + x_0)e^{\lambda t}, y = y_0 e^{\lambda t}$$

5.2.12

(a) By defining $x_1 = I$ and $x_2 = \dot{I}$, we have

$$\dot{x}_1 = x_2, \dot{x}_2 = -\frac{1}{CL}x_1 - \frac{R}{L}x_2$$

(b) The eigenvalues at the origin can be computed by

$$\det(\mathbf{A} - \lambda \mathbf{I})|_{(0,0)} = \det \left[\begin{array}{cc} -\lambda & 1 \\ -\frac{1}{CL} & -\frac{R}{L} - \lambda \end{array} \right] \Bigg|_{(0,0)} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{CL} = 0$$

and the solutions are

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{CL}}\right)$$

If $R > 0$, we have $\Re\lambda_{1,2} < 0$, and the origin is asymptotic stable. If $R = 0$, $\Re\lambda_{1,2} = 0$ and it is neutrally stable.

(c) If $R = 0$, $\lambda_{1,2} = \pm i/\sqrt{CL}$ and it is a center.

If $R > 0$, and

- if $R^2C - 4L > 0$, $\lambda_{1,2}$ are negative real numbers and it is a stable node.
- if $R^2C - 4L = 0$, $\lambda_{1,2} = -R/2L$ and it is a degenerate node.
- if $R^2C - 4L < 0$, $\Re\lambda_{1,2} < 0$ but $\Im\lambda_{1,2} \neq 0$, so it is a stable spiral.