Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 03

February 11, 2015

3 Bifurcations

3.1 Saddle-Node Bifurcation

3.1.1

The saddle-node bifurcation requires that $f(x) = 1 + rx + x^2$ has two identical solutions, so the discriminant $\Delta = r^2 - 4 = 0$. Therefore, $r = \pm 2$ and $x^* = \mp 1$.

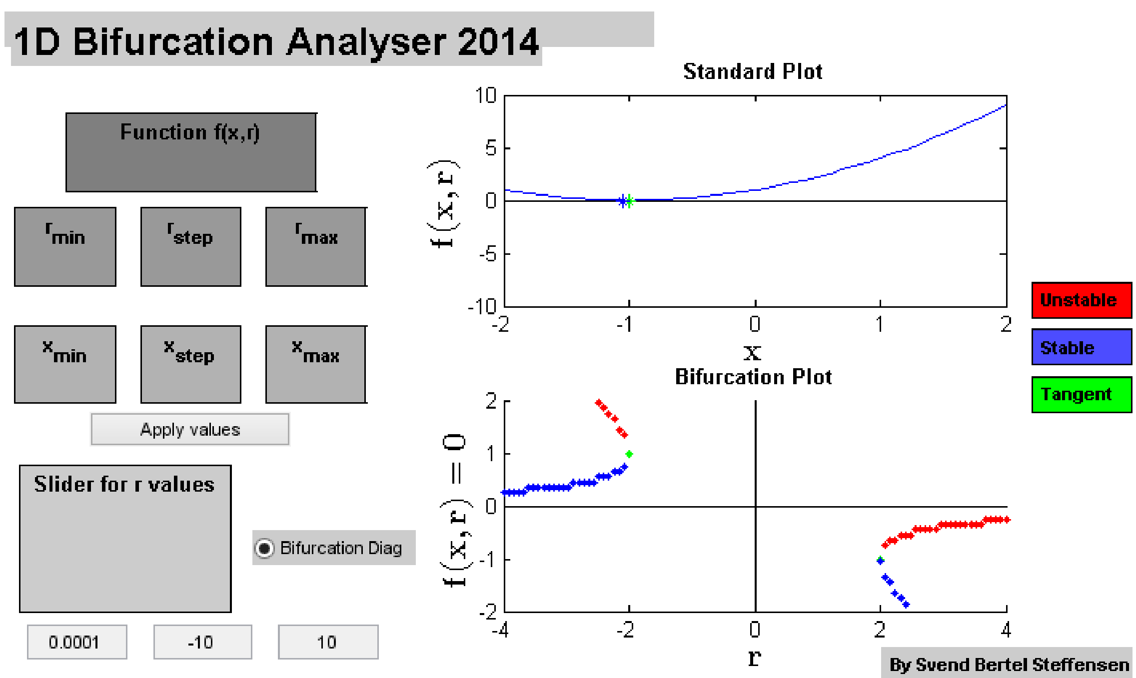

Figure 1: 3.1.1

3.1.3

Since the line $x + r$ are tangent to $\ln(1 + x)$ at the saddle-node bifurcation point, we should solve

$$\frac{\mathrm{d}(x+r)}{\mathrm{d}x} = \frac{\mathrm{d}\ln(1+x)}{\mathrm{d}x}$$
$$x + r = \ln(1 + x)$$

It is easy to show that the solution is $r = 0, x^* = 0$.

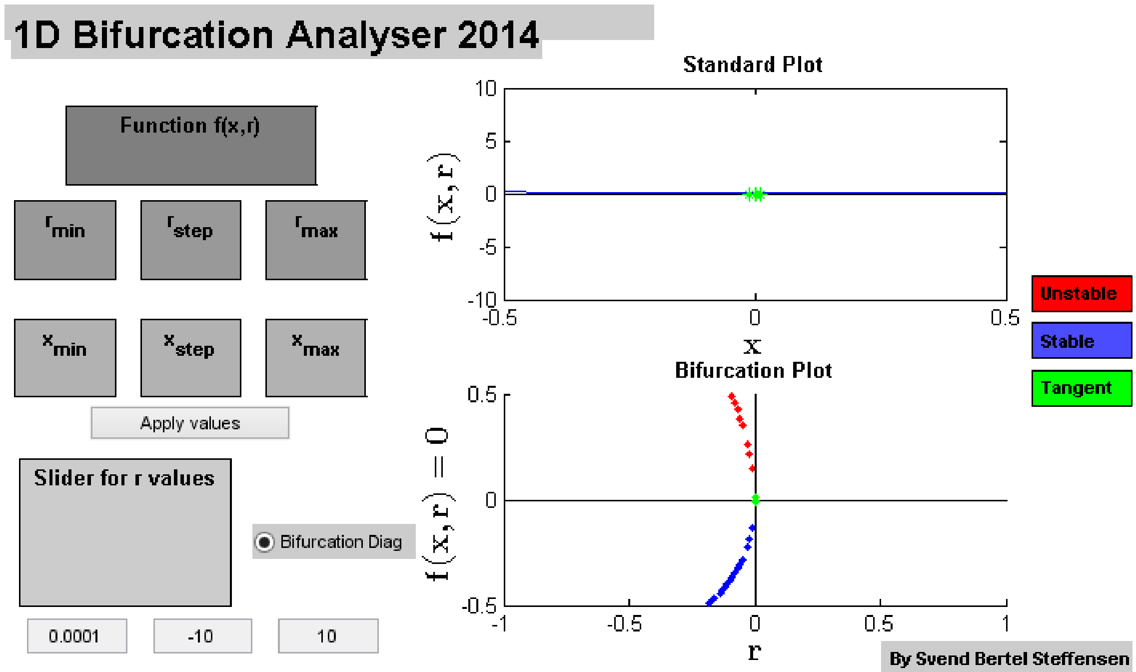

Figure 2: 3.1.3

3.2 Transcritical Bifurcation

3.2.2

$x^* = 0$ is a fix point regardless of the value of r. At transcritical bifurcation, we have $f'(x^*) = 0$. Therefore, $r = 1$.

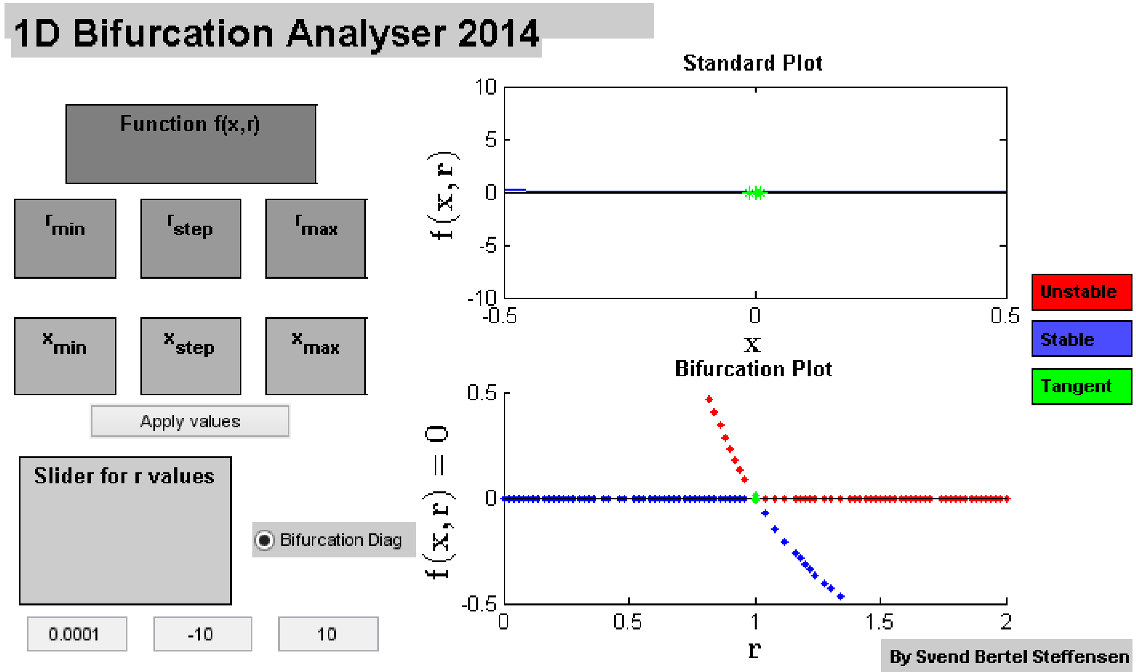

Figure 3: 3.2.2

3.2.4

$x^* = 0$ is always a fix point. Solving $f'(x^*) = 0$ gives $r = 1$.

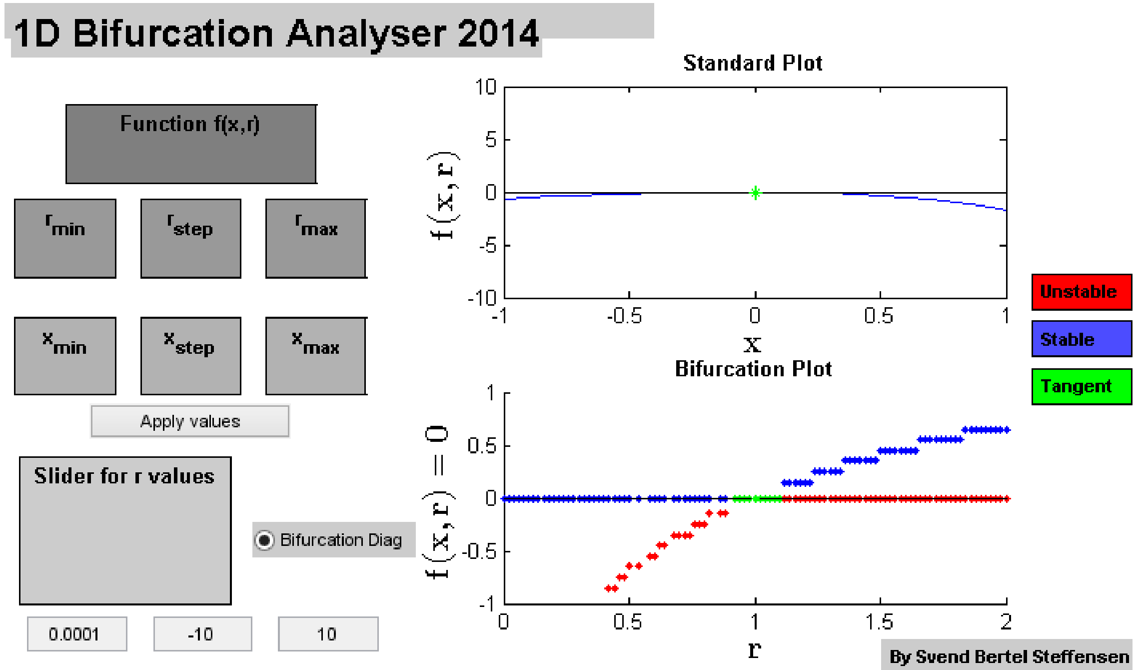

Figure 4: 3.2.4

3.2.5

(a) The rates for the three reactions are

- $A + X \to 2X$: $r_1 = k_1 a x$
- $2X \to A + X$: $r_{-1} = k_{-1} x^2$
- $X + B \to C$: $r_2 = k_2 b x$

Therefore, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r_1 - r_{-1} - r_2 = (k_1 a - k_2 b)x - k_{-1} x^2$$

and $c_1 = k_1 a - k_2 b$, $c_2 = k_{-1}$.

(b) Obviously $x^* = 0$ is a fix point. Since $f'(x^*) = c_1 - 2c_2 x^* = c_1$, if $x^* = 0$ is stable, we should have $k_1 a < k_2 b$. This makes sense since the consumption rate of X is greater than the production rate, so it is not possible to maintain a certain level of X.

3.4 Pitchfork Bifurcation

3.4.1

The bifurcation point requires that $f(x) = 0$ has three identical solutions. Since $x^* = 0$ is already a fix point, we should have that all three solutions of $f(x) = 0$ are 0. Therefore $r = 0$.

When $r > 0$, there is only 1 fix point ($x^* = 0$). Since $f'(x^*) = r > 0$, it is unstable. Therefore, it is subcritical.

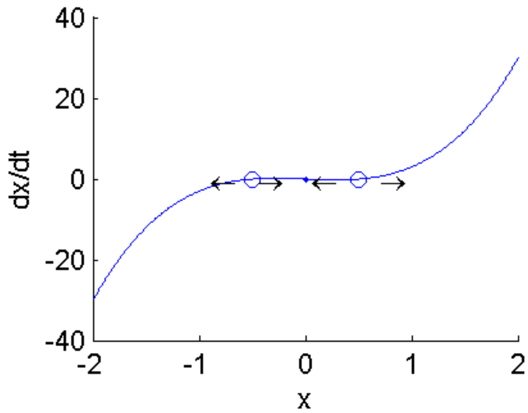

Figure 5: 3.4.1. $r = -1$

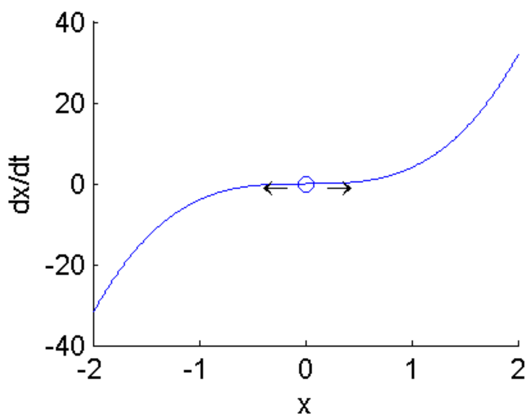

Figure 6: 3.4.1. $r = 0$

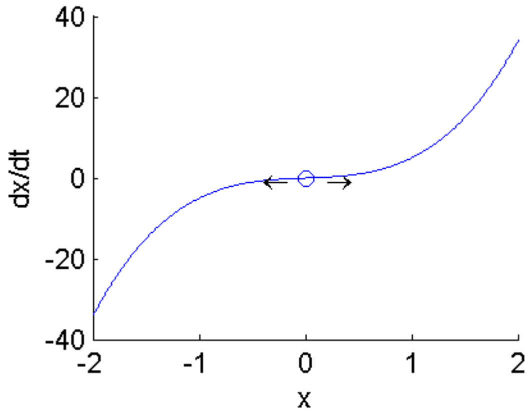

Figure 7: 3.4.1. $r = 1$

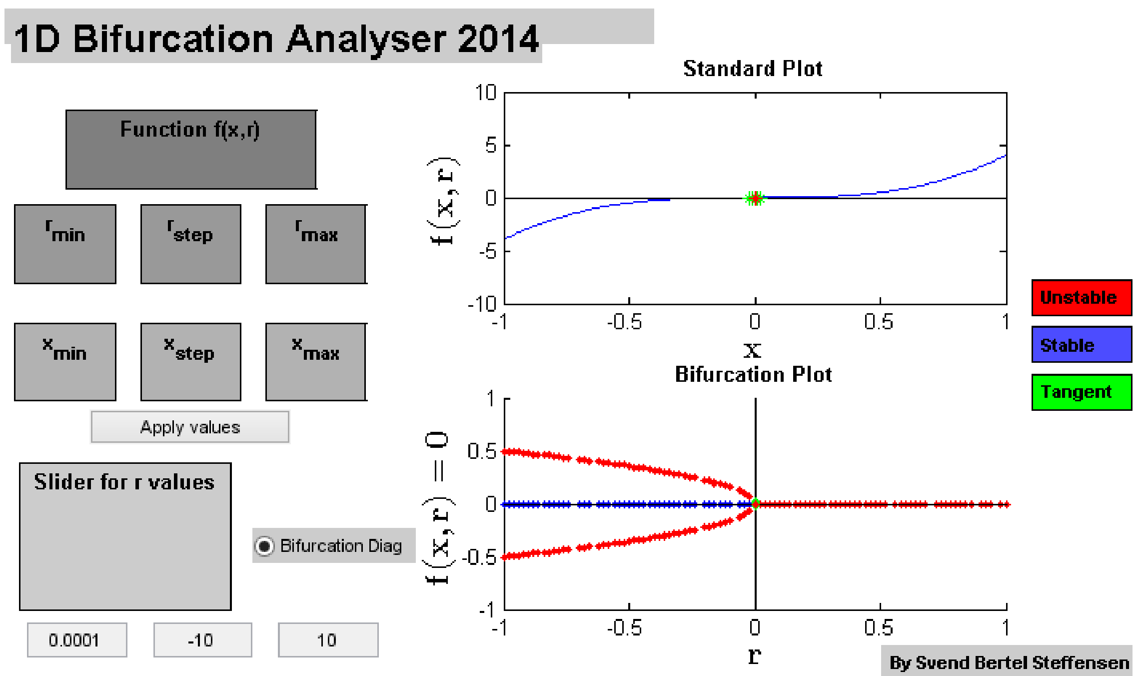

Figure 8: 3.4.1. Bifurcation

3.4.3

Similar to 3.4.1, we have $r = 0$. When $r < 0$, there is only one fix point ($x^* = 0$) and $f'(x^*) = r < 0$. So it is supercritical.

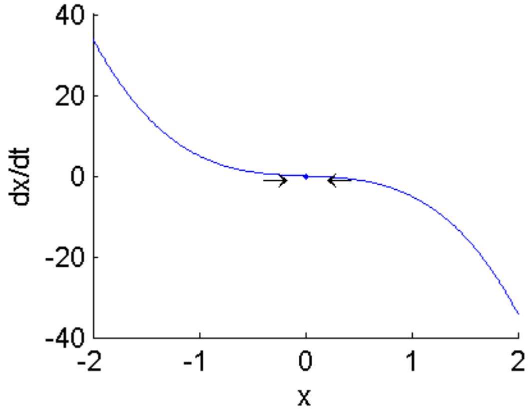

Figure 9: 3.4.3. $r = -1$

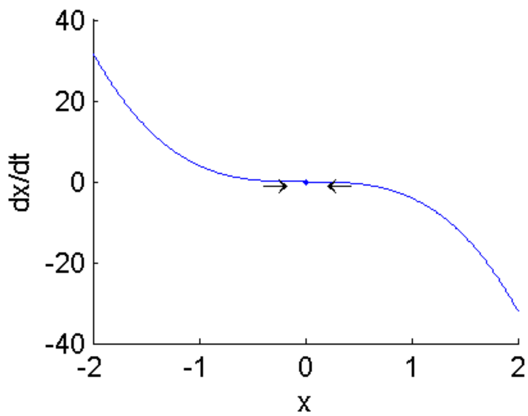

Figure 10: 3.4.3. $r = 0$

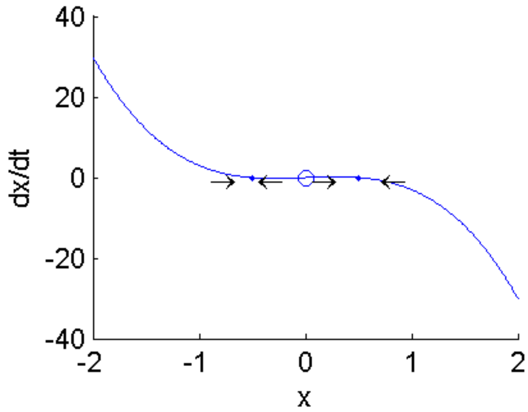

Figure 11: 3.4.3. $r = 1$

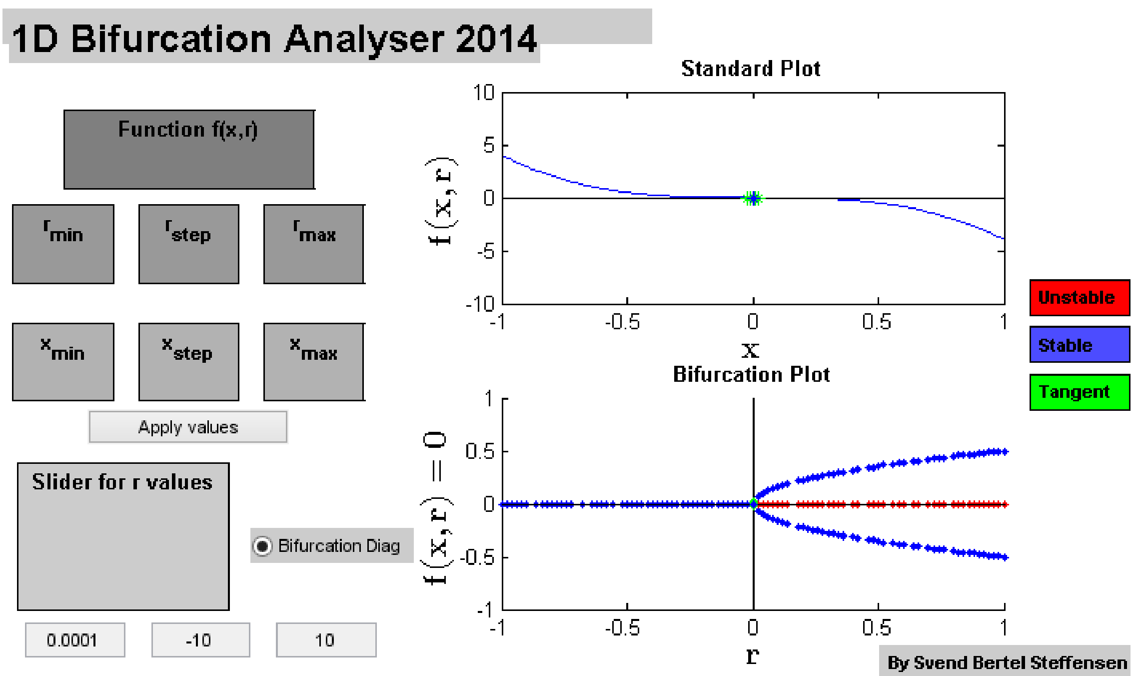

Figure 12: 3.4.3. Bifurcation

3.4.14

(a) By solving $f(x) = 0$, we have

$$x_1^* = 0, x_2^* = \sqrt{\frac{1+\sqrt{1+4r}}{2}}, x_3^* = \sqrt{\frac{1-\sqrt{1+4r}}{2}}, x_4^* = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}, x_5^* = -\sqrt{\frac{1-\sqrt{1+4r}}{2}}$$

$x_2^* - x_5^*$ exist if $r \geq -1/4$. x_3^* and x_5^* exist if $r \leq 0$.

(b)

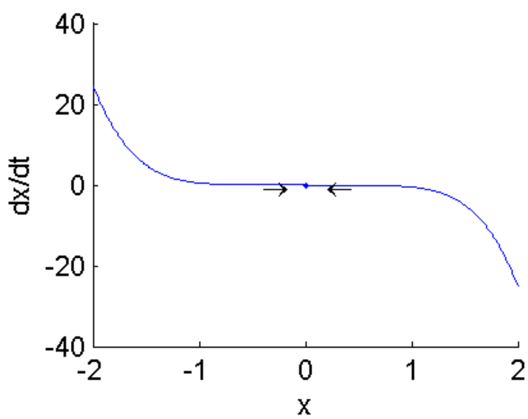

Figure 13: 3.4.14. $r = -1/2$

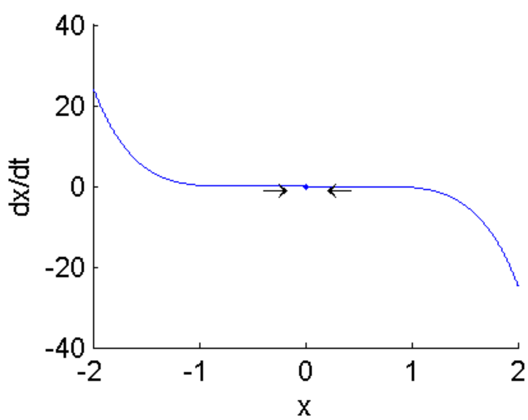

Figure 14: 3.4.14. $r = -1/4$. Only one fix point is found due to numerical issues.

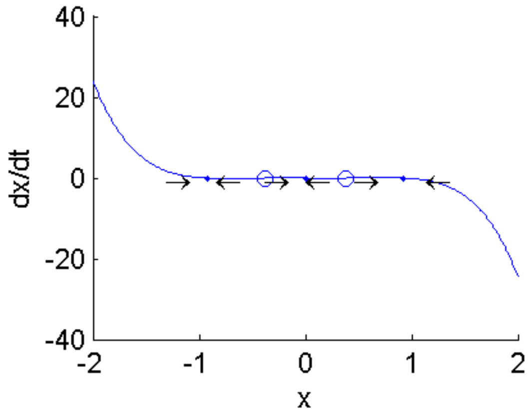

Figure 15: 3.4.14. $r = -1/8$

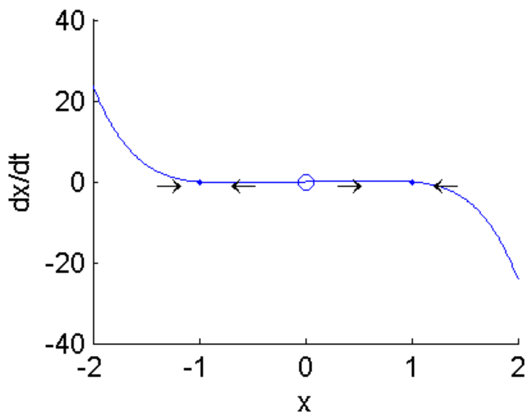

Figure 16: 3.4.14. $r = 0$.

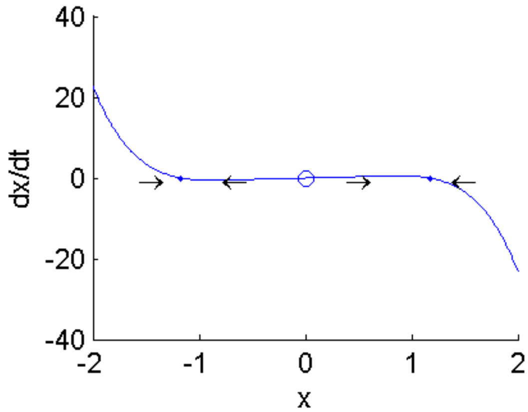

Figure 17: 3.4.14. $r = 1/2$

(c) r_s satisfies that $x_2^* = x_3^*$ and $x_4^* = x_5^*$. Therefore, $r_s = -1/4$.

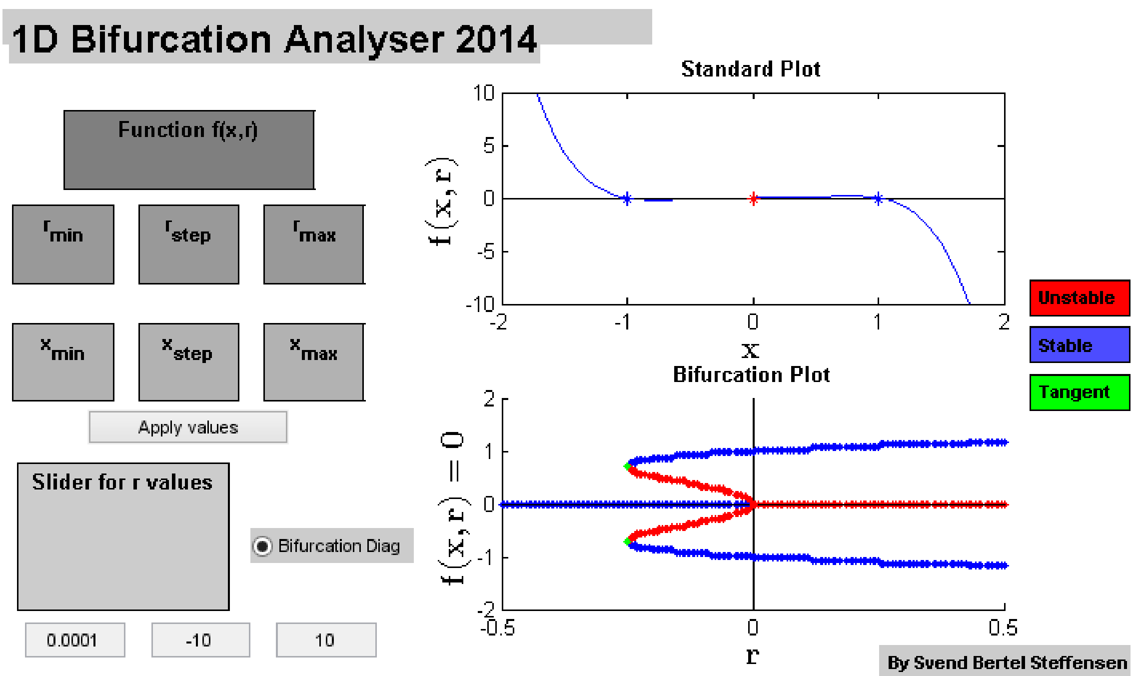

Figure 18: 3.4.14. Bifurcation

3.5 Overdamped Bead on a Rotating Hoop

3.5.8

Plug in $u = xU$ and $t = \tau T$ and we have

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{T}{U}\left(axU + bx^3U^3 - cx^5U^5\right) = aTx + bTU^2x^3 - cTU^4x^5$$

Therefore, we have

$$r = aT, \quad 1 = bTU^2, \quad 1 = cTU^4$$

The solutions are

$$r = \frac{ac}{b^2}, \quad U = \sqrt{\frac{b}{c}}, \quad T = \frac{c}{b^2}$$