Flows on the Line

2.4 Linear Stability Analysis

Let's consider the non-linear first-order equation

$$\frac{dx}{dt} = f(x) \tag{1}$$

with x^* a fixed point of $f(x)$. Then, if η is a small perturbation, such that $x = x^* + \eta$, the Taylor expansion (to second order) of Eq. 1 reads

$$\frac{d(x^* + \eta)}{dt} = \frac{d\eta}{dt} = f(x^*) + f'(x^*)\eta + \frac{1}{2}f''(x^*)\eta^2 + O(\eta^3). \tag{2}$$

Because $f(x^*) = 0$ we have

$$\dot{\eta} \simeq f'(x^*)\eta + \frac{1}{2}f''(x^*)\eta^2. \tag{3}$$

We now find the solution of equation Eq. (3) neglecting the quadratic term in η, thus

$$\begin{aligned} \int \frac{d\eta}{\eta} &= \int f'(x^*)dt + c \\ \eta &= c_1 e^{f'(x^*)t} \end{aligned} \tag{4}$$

Eq. (4) says that any perturbation close a fixed point will decay or grow exponentially, depending on the sign of $f'(x^*)$. A stable solution will always decay to the fixed point, i.e. $f'(x^*) < 0$. If $f'(x^*) = 0$, Eq. (4) is non longer valid to assess the stability of the point x^* and we need to consider higher orders of the Taylor expansion, that is $\sim f^n(x^*)\eta^n$, with $f^n(x^*) \neq 0$.

2.4.1 $\dot{x} = x(1 - x)$

Fixed points are $x = 0$ and $x = 1$ and

$$f'(x) = 1 - 2x, \tag{5}$$

then, because $f'(0) > 0$ and $f'(1) < 0$, $x = 0$ is unstable and $x = 1$ is stable.

2.4.4 $\dot{x} = x^2(6 - x)$

Fixed points are $x = 0$ and $x = 6$, and

$$f'(x) = 12x - 3x^2, \tag{6}$$

then, because $f'(0) = 0$ and $f'(6) < 0$, it follows that $x = 6$ is stable, but we need to use more information to study the stability of $x = 0$, in this case we will use a graphical argument. From Figure 1 we conclude that $x = 0$ is a half-stable fixed point.

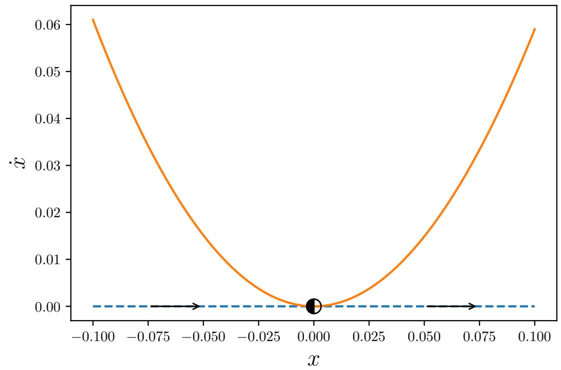

Figure 1: Stability of $x = 0$ for $\dot{x} = x^2(6 - x)$.

2.4.7 $\dot{x} = ax - x^3$

Fixed points depend on the value of a.

- Case (a): $a > 0$
- Case (b): $a < 0$
- Case (c): $a = 0$

with

$$f'(x) = a - 3x^2, \tag{7}$$

Case (a):
Fixed points are $x = 0$ and $x = \pm\sqrt{a}$. Because $f'(0) > 0$ and $f'(\pm\sqrt{a}) < 0$, $x = 0$ is unstable and $x = \pm\sqrt{a}$ are stable.
Case (b):
Fixed point $x = 0$ ($x = \pm\sqrt{a}$ are imaginary roots). In this case $f'(0) < 0$ and $x = 0$ is stable.
Case (c):
Fixed point $x = 0$. In this case $f'(0) = 0$, so we use a graphical argument to study the stability of $x = 0$. From Figure 2 we conclude that $x = 0$ is a stable fixed point.

2.5 Existence and Uniqueness

2.5.3 $\dot{x} = rx + x^3$

In this case, $f(x) = rx + x^3$ and $f'(x) = r + 3x^2$ are continuous in $\mathbb{R}$ for all r in $\mathbb{R}$. Therefore, the solution exist and is unique for any initial condition x_0.

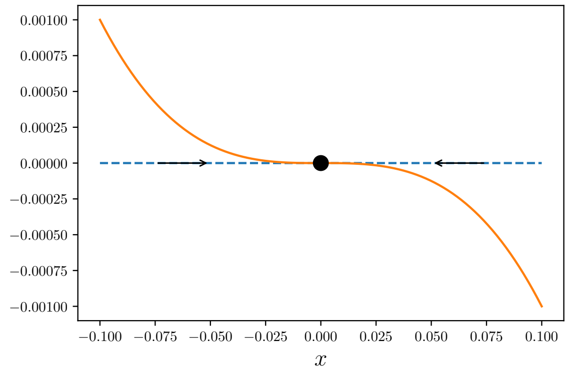

Figure 2: Stability of $x = 0$ for $\dot{x} = -x^3$.

Analytical solution $x(t)$, with $x(0) = x_0 \neq 0$ and $r > 0$:

$$\begin{aligned}
\frac{dx}{x(r+x^2)} &= dt \\
\int \frac{dx}{x(r+x^2)} &= \int dt + c \\
-\frac{1}{2}\ln|r+x^2| + \ln|x| &= rt + c_1 \\
\frac{x}{\sqrt{r+x^2}} &= e^{c_1}e^{rt} \\
\frac{x^2}{r+x^2} &= c_2 e^{2rt} \\
x(t) &= \pm\sqrt{\frac{rc_2e^{r2t}}{1-c_2e^{2rt}}}
\end{aligned} \tag{8}$$

with $c_2 = x_0^2/(r + x_0^2)$.

Then, the solution $x(t) \to \pm\infty$ when $1 - x_0^2/(r + x_0^2)e^{2rt} = 0$, this happen for $t = \ln\left(\frac{x_0^2+r}{x_0^2}\right)\frac{1}{2r}$.

2.5.4 $\dot{x} = x^{1/3}$ has infinite solutions

In this case, $f(x) = x^{1/3}$ is continuous and $f'(x) = \frac{1}{3}x^{-2/3}$ is discontinuous at $x = 0$. Because f is continuous, the solution exist, but we can not ensure that the solution is unique.

Analytical solution $x(t)$ with $x(0) = 0$:

$$\begin{aligned}\frac{dx}{x^{1/3}} &= dt \\ \int \frac{dx}{x^{1/3}} &= \int dt + c \\ \frac{3}{2}x^{2/3} &= t + c \\ x &= \pm\left[\frac{2}{3}(t+c)\right]^{3/2}\end{aligned} \tag{9}$$

The initial condition is satisfied if $c = 0$, so $x(t) = \left(\frac{2}{3}t\right)^{3/2}$, with $t \geq 0$. On the other hand, the function $x = -\left(\frac{2}{3}t\right)^{3/2}$ is also a solution of the initial value problem. Moreover, the function $x = 0$ for $t \geq 0$ is yet another solution.

Finally, we can construct a family of solutions, for any arbitrary positive t_0, of the form:

$$x(t) = \begin{cases} 0, & \text{if } 0 \leq t < t_0 \\ \pm[\frac{2}{3}(t - t_0)]^{3/2}, & \text{if } t \geq t_0 \end{cases}$$

which are continuous and differentiable (in particular in $t = t_0$).

2.6 Impossibility of the Oscillations

2.6.2

Let's consider the first order initial value problem defined by $\dot{x} = f(x)$, with f a continuous function in $[x(t+T), x(t)]$. Let's assume that x is a periodic function of t, such that $x(t+T) = x(t)$, with $T > 0$. We want to prove that the only possible periodic function $x(t)$, that is solution of $\dot{x} = f(x)$, is the constant function.

First, we have that the integral

$$\int_t^{t+T} f(x)\frac{dx}{dt}dt = \int_t^{t+T} \left(\frac{dx}{dt}\right)^2 dt \geq 0, \tag{10}$$

which can be proven using the Cauchy-Schwarz Inequality

$$\left[\int_a^b h(t)g(t)dt\right]^2 \leq \int_a^b h(t)^2 dx \int_a^b g(t)^2 dt \tag{11}$$

taking $h(t) = dx/dt$, $g(t) = 1$, $a = t$ and $b = t+T$. Because x is periodic in $[t, t+T]$ it follows that dx/dt is periodic in $[t, t+T]$, and furthermore $0 \leq [\int_t^{t+T} \frac{dx}{dt}\, dt]^2 = [\int_a^b h(t)g(t)dt]^2$. Thus, Eq. (11) transforms in

$$0 \leq \int_a^b h(t)^2 dx \int_a^b g(t)^2 dt = (b-a) \int_a^b h(t)^2 dx \tag{12}$$

where we used that $\int_a^b g(t)^2 dt = \int_a^b 1 dt = b - a = T$.
Going back to the integral of Eq. (10)

$$\int_t^{t+T} f(x) \frac{dx}{dt} dt = \int_{x(t)}^{x(t+T)} f(x) dx = 0, \tag{13}$$

because $x(t) = x(t+T)$.
Finally, we have that

$$0 = \int_{x(t)}^{x(t+T)} f(x) dx = \int_t^{t+T} f(x) \frac{dx}{dt} dt = \int_t^{t+T} \left(\frac{dx}{dt}\right)^2 dt \geq 0 \tag{14}$$

which will be only valid when $\int_t^{t+T} \left(\frac{dx}{dt}\right)^2 dt = 0$ if and only if $dx/dt = 0$, and thus x is necessarily a constant function of t.

2.7 Potentials

2.7.6

The potential, V, satisfies

$$-\frac{dV}{dx} = r + x - x^3, \tag{15}$$

integrating this equations, and furthermore assuming the constant of integration $c = 0$, we obtain

$$V(x) = -rx - \frac{1}{2}x^2 + \frac{1}{4}x^4 \tag{16}$$

The fixed points are the solution of $0 = r + x - x^3$, which depends on the value of r. The discriminant of this cubic equation is $\Delta = r^2/4 - 1/27$, the we have two cases:

- $\Delta < 0$: 3 fixed points.
- $\Delta \geq 0$: 1 fixed point.

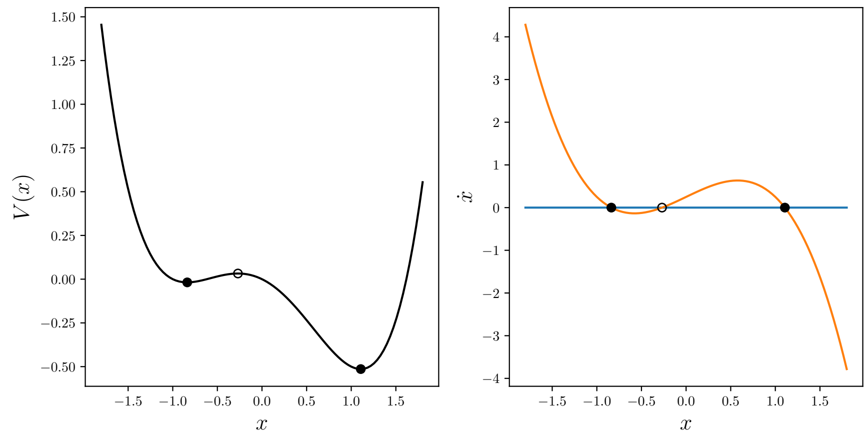

Figure 3: Case $\Delta < 0$ with $r = 1/4$. Left panel: Potential $V(x)$. Right panel: $\dot{x}$. The locals minimum of the potential correspond to the stable fixed points, while the local maximum corresponds to the unstable fixed point.

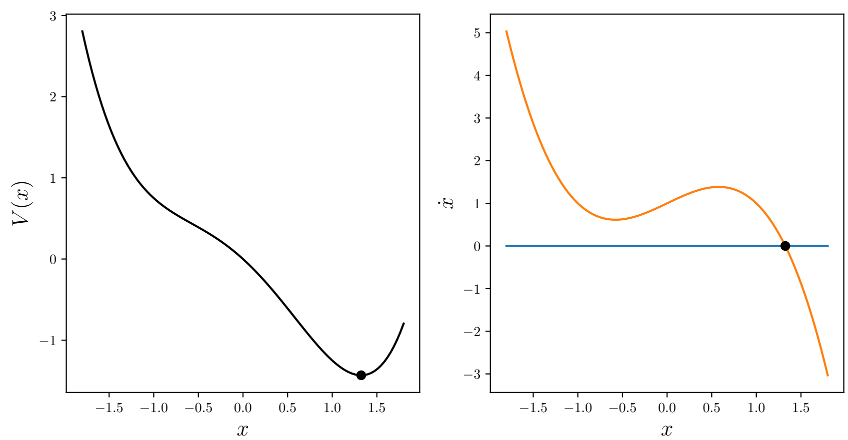

Figure 4: Case $\Delta > 0$ with $r = 1$. Left panel: Potential $V(x)$. Right panel: $\dot{x}$. The locals minimum of the potential correspond to the stable fixed points, while the local maximum corresponds to the unstable fixed point.