Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 2

February 8, 2015

2 Flows on the Line

2.4 Linear Stability Analysis

2.4.1

By solving $f(x^*) = 0$, we have $x^* = 0, 1$. Note that $f'(x) = 1 - 2x$, and

- For $x^* = 0$, $f'(x^*) = 1 > 0$ and the fix point is unstable.
- For $x^* = 1$, $f'(x^*) = -1 < 0$ and the fix point is stable.

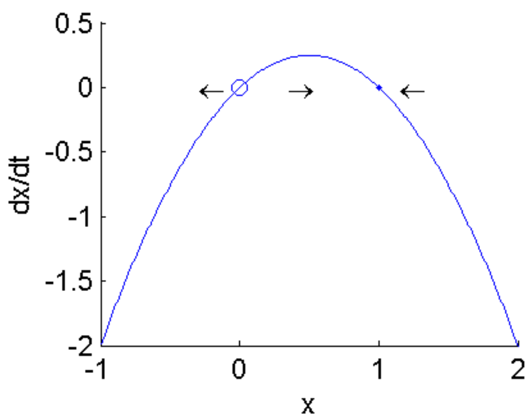

Figure 1: 2.4.1

2.4.4

The fix points are $x^* = 0, 6$ and the derivate is $f'(x) = 12x - 3x^2$. For $x^* = 6$, we have $f'(x^*) = -36 < 0$ and it is a stable fix point. For $x^* = 0$, $f'(x^*) = 0$ and we cannot determine its stability directly.

We then define $x = x^* + \eta$ where $x^* = 0$ and the infinitesimal perturbation $\eta \ll 1$. The evolution of η follows

$$\dot{\eta} = f(x^* + \eta) - f(x^*) = f'(x^*)\eta + \frac{1}{2}f''(x^*)\eta^2 + o(\eta^2) = (6 - 3x^*)\eta^2 + o(\eta^2) = 6\eta^2 + o(\eta^2)$$

When the perturbation is negative ($\eta < 0$), the system is able to evolve back to the steady state; when $\eta > 0$, the system will not be able to evolve back. Therefore, $x^* = 0$ is unstable.

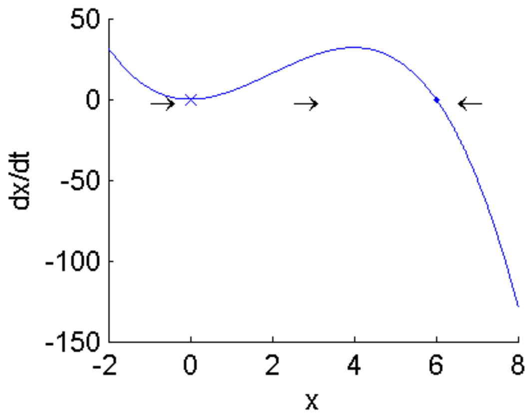

Figure 2: 2.4.4

2.4.7

(a) If $a > 0$, the fix points are $x^* = 0, -\sqrt{a}, \sqrt{a}$. We have $f'(x) = a - 3x^2$. For $x^* = 0$, $f'(x^*) = a > 0$ and it is an unstable fix point. For $x^* = \pm\sqrt{a}$, $f'(x^*) = -2a < 0$ and they are stable fix points.

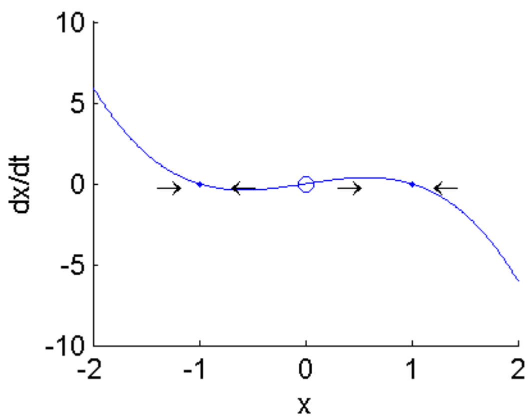

Figure 3: 2.4.7, $a = 1$

(b) If $a = 0$, the fix point is $x^* = 0$. Since $f'(x^*) = 0$, we need to expand f to higher order. Denote the perturbation as η and we have

$$\dot{\eta} = -6\eta^3 = (-6\eta^2)\eta$$

$-6\eta^2 < 0$ and the fix point $x^* = 0$ is stable.

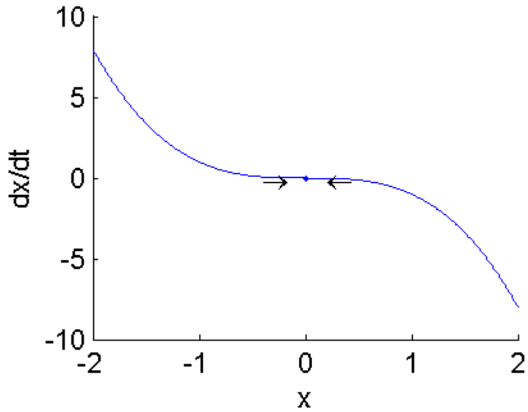

Figure 4: 2.4.7, $a = 0$

(c) If $a < 0$, the fix point is $x^* = 0$. Since $f'(x^*) = a < 0$, it is stable.

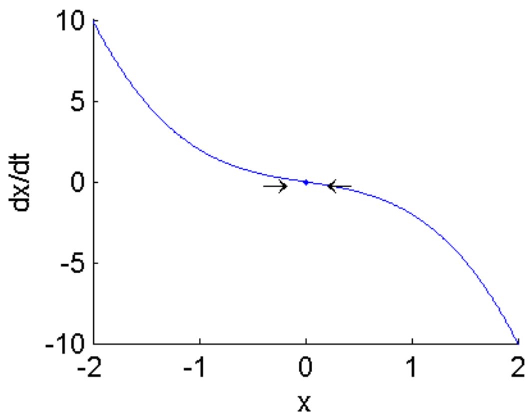

Figure 5: 2.4.7, $a = -1$

2.5 Existence and Uniqueness

2.5.3

Firstly, $f(x) = rx + x^3$ and $f'(x) = r + 3x^2$ are continuous in $\mathbb{R}$. Therefore, given an initial state, there exists one solution and the solution is unique.

To prove that the x reaches infinity within finite time, we need to solve x explicitly. Following a similar method in question 2.2, we have

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = rx + x^3 \\
&\frac{\mathrm{d}x}{rx + x^3} = \mathrm{d}t \\
&\frac{1}{r}\left(\frac{1}{x} - \frac{1}{2}\frac{1}{x + i\sqrt{r}} - \frac{1}{2}\frac{1}{x - i\sqrt{r}}\right)\mathrm{d}x = \mathrm{d}t \\
&\frac{1}{r}\mathrm{d}\ln\left|\frac{x}{\sqrt{x^2 + r}}\right| = \mathrm{d}t \\
&\ln\left|\frac{x}{\sqrt{x^2 + r}}\right| - \ln\left|\frac{x_0}{\sqrt{x_0^2 + r}}\right| = rt
\end{aligned}$$

When $x \to \pm\infty$, $\ln|x/\sqrt{x^2 + r}| \to 0$. Therefore we have

$$-\ln\left|\frac{x_0}{\sqrt{x_0^2 + r}}\right| = rt_\infty$$

Note that when $|x_0| < \infty$, the term $-\infty < \ln|x_0/\sqrt{x_0^2 + r}| < 0$. We then have $0 < t_\infty < \infty$, indicating that x can evolve to infinity within finite time.

2.5.4

Assume that x takes off at time $t_0 \neq 0$, and we have

$$\dot{x} = x^{1/3} \to x^{-1/3}\mathrm{d}x = \mathrm{d}t \to \frac{3}{2}\mathrm{d}x^{2/3} = \mathrm{d}t \to x = \begin{cases} 0 & t < t_0 \\ \left[\frac{2}{3}(t - t_0)\right]^{3/2} & t \geq t_0 \end{cases}$$

Since t_0 can be any non-negative value, there are infinite solutions of x.

2.6 Impossibility of Oscillations

2.6.2

On one hand,

$$\int_t^{t+T} f(x)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \int_{x(t)}^{x(t+T)} f(x)\mathrm{d}x = 0$$

On the other hand,

$$\int_t^{t+T} f(x)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \int_t^{t+T} f(x)^2\mathrm{d}t \geq 0$$

The "=" is valid if and only if $\forall t^* \in (t, t+T)$, $f(x(t^*)) = 0$. However, this leads to a trivial solution and doesn't count as a periodic solution. Therefore, it is impossible to have oscillatory solutions for 1D system.

2.7 Potentials

2.7.6

The potential is

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -r - x + x^3 \rightarrow V = C - rx - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

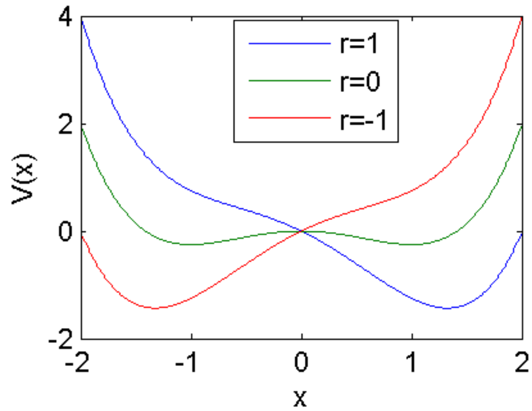

Figure 6: 2.7.6, Potential $V(x)$. Assume the constant is 0.

The fix points can be obtained by solving $-\mathrm{d}V/\mathrm{d}x = r + x - x^3 = 0$. The discriminant is $\Delta = r^2/4 - 1/27$.

- If $\Delta < 0$, there are 3 fix points. From low to high, they are stable, unstable and stable, respectively.

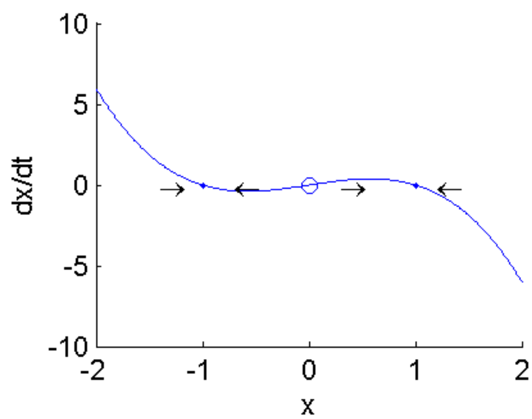

Figure 7: 2.7.6, $r = 0$.

- If $\Delta \geq 0$, there is only 1 fix point, and it is stable.

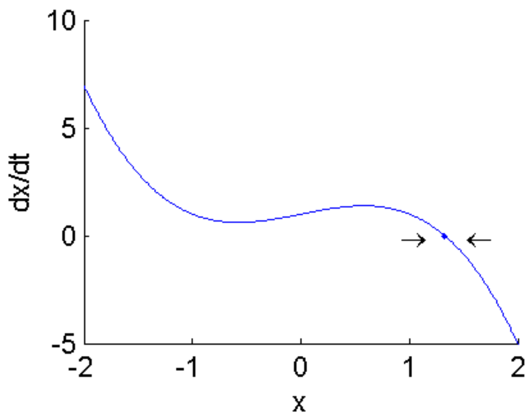

Figure 8: 2.7.6, $r = 1$.

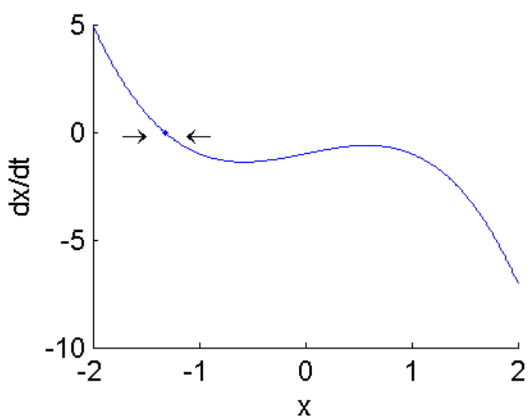

Figure 9: 2.7.6, $r = -1$.