Flows on the Line

2.1 A Geometric Way of Thinking

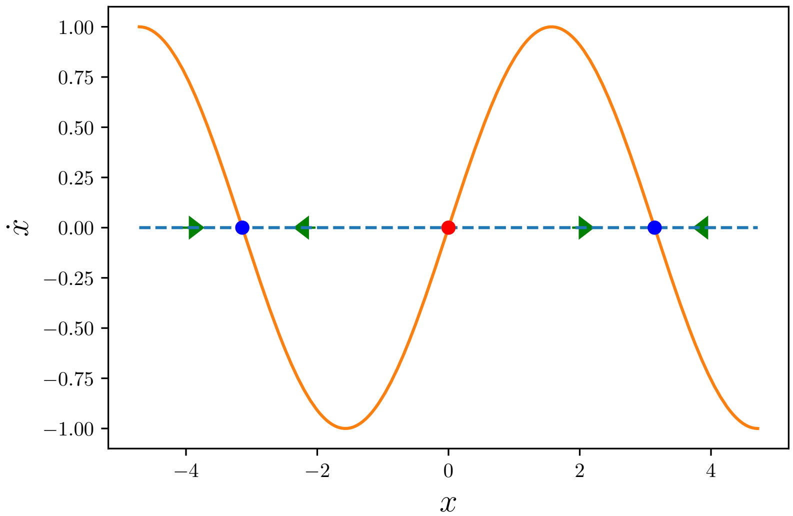

Figure 1: Plot of $\dot{x} = \sin x$. The red circles are the unstable fixed points. The blue circles are the stable fixed points. The green arrows indicate the velocity-field direction between the fixed points.

2.1.1 Fixed points

$$\dot{x} = \sin x = 0 \to x = k\pi, k \in \mathbb{Z} \tag{1}$$

2.1.2 Greatest velocity to the right

$$\sin x = 1 \to x = 2k\pi + \frac{\pi}{2}, k \in \mathbb{Z} \tag{2}$$

2.1.3 Acceleration

Flow acceleration as a function of x

$$\ddot{x} = (\cos x)\dot{x} = \cos x \sin x = \frac{1}{2}\sin(2x) \tag{3}$$

Points where the flow has a maximum positive acceleration

$$\frac{1}{2}\sin(2x) = \frac{1}{2} \to x = k\pi + \frac{\pi}{4}, k \in \mathbb{Z} \tag{4}$$

2.1.4 Analytical solution $x(t)$

Step 0: Write $\csc x + \cot x = \cot \dfrac{x}{2}$. Using the following two properties

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \tag{5}$$

and

$$\cos x = 2 \cos^2 \frac{x}{2} - 1 = 1 - 2 \sin^2 \frac{x}{2}, \tag{6}$$

we then have

$$\csc x + \cot x = \frac{1}{\sin x} + \frac{\cos x}{\sin x} = \frac{2 \cos^2 (x/2)}{2 \sin(x/2) \cos(x/2)} = \cot \frac{x}{2}. \tag{7}$$

Step1: Use this result and replace it in the expression for the time t,

$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right| = \ln \left| \frac{\cot (x_0/2)}{\cot(x/2)} \right| = \ln \left| \frac{\tan(x/2)}{\tan (x_0/2)} \right| \tag{8}$$

where we are always assuming that $x_0 \neq k\pi, k \in \mathbb{Z}$, i.e. the fixed points of $\dot{x} = \sin x$. Taking the exponential to both sides we obtain

$$\tan \frac{x}{2} = \mathrm{e}^t \tan \frac{x_0}{2}. \tag{9}$$

Finally,

$$x(t) = 2 \tan^{-1} \left(\mathrm{e}^t \tan \frac{x_0}{2} \right) \tag{10}$$

Now, setting $x_0 = \frac{\pi}{4}$ we obtain

$$\tan \frac{x_0}{2} = \frac{1}{\csc x_0 + \cot x_0} = \frac{1}{\sqrt{2} + 1} \tag{11}$$

$$x = 2 \tan^{-1} \left(\frac{\mathrm{e}^t}{1 + \sqrt{2}} \right) \tag{12}$$

2.2 Fixed points and stability

2.2.1 $\dot{x} = 4x^2 - 16$

The fixed points are: $x = -2$, stable and $x = 2$, unstable.

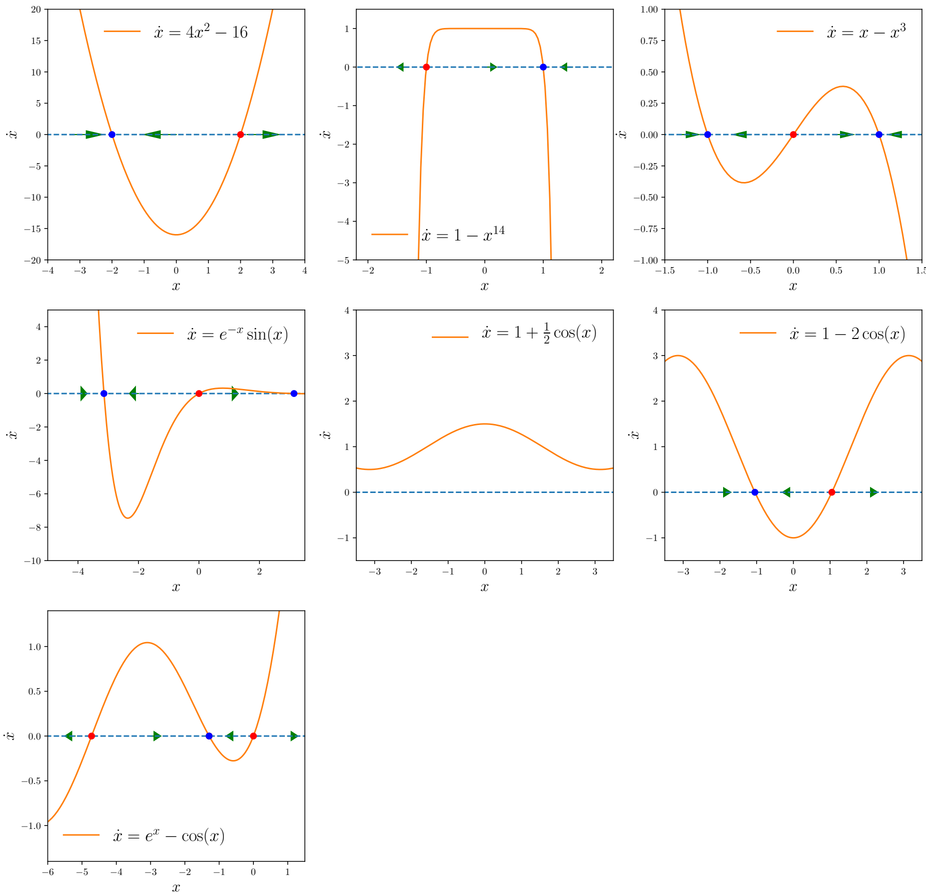

Figure 2: Plots of the different functions $\dot{x} = f(x)$. The red circles are the unstable fixed points. The blue circles are the stable fixed points. The green arrows indicate the velocity-field direction between the fixed points.

Analytical solution $x(t)$ for $x \neq \pm 2$:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 4x^2 - 16 \\ \int \frac{\mathrm{d}x}{4x^2 - 16} &= \int \mathrm{d}t + c \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{1}{4}\int \frac{\mathrm{d}x}{x-2} - \frac{1}{4}\int \frac{\mathrm{d}x}{x+2} &= 4t + c \\ \ln\left|\frac{x-2}{x+2}\right| &= 16t + c \\ \frac{x-2}{x+2} &= e^{16t}e^{c} = Ce^{16t}. \end{aligned} \tag{14}$$

Thus, $x(t)$ can be obtaining by

$$\begin{aligned} x - 2 &= (x+2)Ce^{16t} \\ x(1 - Ce^{16t}) &= 2(1 + Ce^{16t}) \\ x &= 2\frac{1 + Ce^{16t}}{1 - Ce^{16t}} \end{aligned} \tag{15}$$

Using the initial value $x(t=0) = x_0$, we determine the value of constant C,

$$\frac{x_0 - 2}{x_0 + 2} = C. \tag{16}$$

2.2.2 $\dot{x} = 1 - x^{14}$

The fixed points are: $x = -1$, unstable and $x = 1$, stable. Note that the equation $1 - x^{14}$ has 14 roots, but only 2 are real. A general expression for the roots, x_f, is given by

$$x_f = e^{i\pi n/7} \text{ for } n = 0, 1, \ldots, 13 \tag{17}$$

where for $n = 0$ and $n = 7$ we obtain the values 1 and -1, respectively.
No analytic solution for $x(t)$.

2.2.3 $\dot{x} = x - x^3$

The fixed points are: $x = -1, 1$, stable and $x = 0$, unstable.
Analytical solution $x(t)$ for $x \neq \pm 1, 0$:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= x - x^3 \\
\frac{\mathrm{d}x}{x - x^3} &= \mathrm{d}t \\
\int \left(\frac{1}{x} + \frac{1}{2}\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x} \right) \mathrm{d}x &= \int \mathrm{d}t + c \\
\ln|x| - \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \ln \frac{|x|}{\sqrt{|x^2-1|}} &= t + c \\
\frac{x}{\sqrt{|x^2-1|}} &= e^t e^c \\
\frac{x^2}{x^2-1} &= Ce^{2t} \\
x^2 &= (x^2-1)Ce^{2t} \\
x^2(-1+Ce^{2t}) &= Ce^{2t} \\
x &= \pm\sqrt{\frac{Ce^{2t}}{Ce^{2t}-1}}.
\end{aligned} \tag{18}
$$

Defining $C_1 = 1/C$ we have

$$
x = \pm \frac{e^t}{\sqrt{e^{2t} - C_1}}. \tag{19}
$$

The constant C_1 can be determined from the initial condition $x(t = 0) = x_0$, then

$$
C_1 = \frac{x_0^2 - 1}{x_0^2} \tag{20}
$$

2.2.4 $\dot{x} = e^{-x}\sin x$

The fixed points are: $x = 2k\pi$, unstable and $x = (2k+1)\pi$, stable, with $k \in \mathbb{Z}$.
No analytic solution for $x(t)$.

2.2.5 $\dot{x} = 1 + \frac{1}{2}\cos x$

No fixed points.
Analytical solution $x(t)$. We first use the following property

$$
\frac{1}{1 + \tan^2 x} = \cos^2 x, \tag{21}
$$

then, defining $u = \tan x/2$ we have

$$1 + \frac{1}{2}\cos x = \frac{1}{2} + \cos^2\frac{x}{2} = \frac{1}{2} + \frac{1}{1+u^2} \tag{22}$$

$$\mathrm{d}x = \frac{2}{1+u^2}\mathrm{d}u \tag{23}$$

using this substitution, the equation reads as follows

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 1 + \frac{1}{2}\cos x \\
\frac{\mathrm{d}x}{1+(1/2)\cos x} &= \mathrm{d}t \\
\frac{1}{1/2 + 1/\left(1+u^2\right)}\frac{2}{1+u^2}\mathrm{d}u &= \mathrm{d}t \\
\int \frac{4\mathrm{d}u}{u^2+3} &= \int \mathrm{d}t + c \\
\frac{4}{3}\int \frac{\mathrm{d}u}{(u/\sqrt{3})^2+1} &= t + c \\
\frac{4}{\sqrt{3}}\tan^{-1}(u/\sqrt{3}) &= t + c \\
u &= \sqrt{3}\tan\left(\frac{\sqrt{3}}{4}t + c_1\right) \\
\tan x/2 &= \sqrt{3}\tan\frac{\sqrt{3}}{4}t + c_1 \\
x &= 2\tan^{-1}\left(\sqrt{3}\tan\left(\frac{\sqrt{3}}{4}t + c_1\right)\right)
\end{aligned} \tag{24}$$

The constant c_1 can be determined from the initial condition $x(t=0) = x_0$, then

$$c_1 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x_0}{2}\right). \tag{25}$$

2.2.6 $\dot{x} = 1 - 2\cos x$

The fixed points are: $x = 2k\pi + \pi/3$, unstable and $x = 2k\pi - \pi/3$, stable, with $k \in \mathbb{Z}$. Once again we define $\tan x/2 = u$ which gives:

$$\cos(x) = \frac{1-u^2}{1+u^2} \quad \mathrm{d}x = 2\frac{\mathrm{d}u}{1+u^2}. \tag{26}$$

$$\begin{aligned} \frac{\mathrm{d}x}{1-2\cos(x)} &= \mathrm{d}t \\ \frac{2}{3u^2-1}\mathrm{d}u &= \mathrm{d}t \\ 2\int \frac{\mathrm{d}u}{(u\sqrt{3})^2-1} &= \int \mathrm{d}t + c_1 \end{aligned} \tag{27}$$

Now, let's define $s = \sqrt{3}u$, then $u = 1/\sqrt{3}s$ and $\mathrm{d}u = \mathrm{d}s/\sqrt{3}$, it follows that

$$\begin{aligned} \frac{2}{\sqrt{3}}\left[\int \frac{\mathrm{d}s}{s-1} - \int \frac{\mathrm{d}s}{s+1}\right] &= t + c_1 \\ \frac{1}{\sqrt{3}}\left[\ln(s-1) - \ln(s+1)\right] &= t + c_1. \end{aligned} \tag{28}$$

Solving for u leads to

$$\begin{aligned} \frac{1}{\sqrt{3}}\ln\left|\frac{u-1/\sqrt{3}}{u+1/\sqrt{3}}\right| &= t + c_1 \\ \frac{u-1/\sqrt{3}}{u+1/\sqrt{3}} &= Ce^{\sqrt{3}t} \\ u &= \frac{1}{\sqrt{3}}\frac{1+Ce^{\sqrt{3}t}}{1-Ce^{\sqrt{3}t}} \\ x = 2\tan^{-1}&\left(\frac{1}{\sqrt{3}}\frac{1+Ce^{\sqrt{3}t}}{1-Ce^{\sqrt{3}t}}\right) \end{aligned} \tag{29}$$

The constant C can be determined from the initial condition $x(t=0) = x_0$, then

$$C = \frac{\tan x_0/2 - 1/\sqrt{3}}{\tan x_0/2 + 1/\sqrt{3}}. \tag{30}$$

2.2.7 $\dot{x} = e^x - \cos x$

By looking at the equation we find that $x = 0$ is a fixed point, solution of the equation $e^x - \cos(x) = 0$. Furthermore $x = 0$ is an unstable point (see Figure 1, bottom panel.)

This equation has not analytical solution for its roots. However, there are different numerical methods that could be used to estimate the value of the roots. For instance, by carefully looking at the plot, we can identify some of the intervals where these roots are bounded, and then apply a simple bisection algorithm.

No analytic solution for $x(t)$.

2.2.11 Analytical solution for a charging capacitor

The equation for the charging capacitor reads

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}, \tag{31}$$

which can be rearranged in terms of the constants $a = -1/RC$ and $b = V_0C$ as follows

$$\dot{Q} = a(Q - b). \tag{32}$$

$$\begin{aligned} \int \frac{\mathrm{Q}}{Q-b} &= a \int \mathrm{d}t + c \\ \ln|Q - b| &= at + c_1 \\ Q &= b + e^{at}e^{c_1} \\ Q &= V_0C + c_2e^{-t/RC} \end{aligned} \tag{33}$$

The initial condition is such that $Q(t = 0) = 0$, then the constant $c_2 = -V_0C$, and the solution reads

$$Q = V_0C\left(1 - e^{-t/RC}\right) \tag{34}$$

2.3 Population Growth

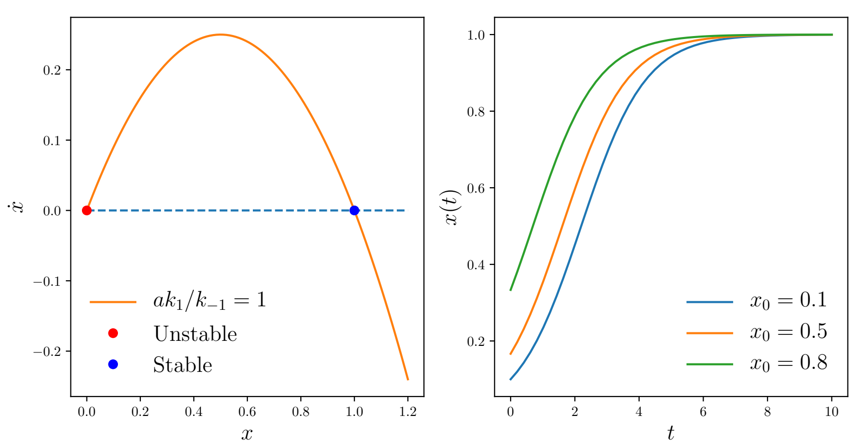

Figure 3: Left panel: Plot of $\dot{x}$ for $a = k_1 = k_{-1} = 1$. The red circles are the unstable fixed points. The blue circles are the stable fixed points. Right panel: Plot of $x(t)$ for different initial conditions x_0.

$$\dot{x} = k_1 a x - k_{-1} x^2 \tag{35}$$

Fixed points are $x = 0$, unstable, and $x = a\frac{k_1}{k_{-1}}$, stable.

$$\begin{aligned}
\frac{dx}{-k_{-1}(x - \frac{k_1 a}{k_{-1}})x} &= dt \\
\int \frac{dx}{(x - \frac{k_1 a}{k_{-1}})x} &= -k_{-1}t + c \\
\frac{-k_{-1}}{k_1 a}\ln x + \frac{k_{-1}}{k_1 a}\ln(x - \frac{k_1 a}{k_{-1}}) &= -k_{-1}t + c \\
\frac{x - \frac{k_1 a}{k_{-1}}}{x} &= e^{-k_1 a t + c} \\
x - \frac{k_1 a}{k_{-1}} &= x e^{-k_1 a t + c} \\
x(1 - e^{-k_1 a t + c}) &= \frac{k_1 a}{k_{-1}}
\end{aligned} \tag{36}$$

The analytical solution is $x(t) = \frac{k_1 a}{k_{-1}}\frac{1}{(1 - c_1 e^{-k_1 a t})}$, with $c_1 = 1 - \frac{k_1 a / k_{-1}}{x_0}$.