Dynamical Systems and Chaos 2015 Spring

Homework Solutions, Session 1

February 4, 2015

2 Flows on the Line

2.1 A Geometric Way of Thinking

2.1.1

$$\dot{x} = \sin x = 0 \rightarrow x = k\pi, k \in \mathbb{Z}$$

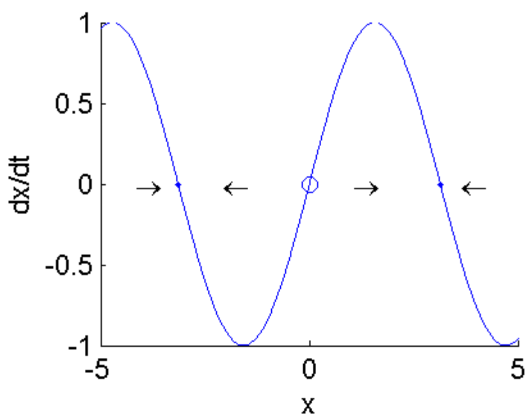

Figure 1: 2.1.1

2.1.2

$$\sin x = 1 \rightarrow x = 2k\pi + \frac{\pi}{2}, k \in \mathbb{Z}$$

2.1.3

(a)

$$\ddot{x} = (\cos x)\dot{x} = \cos x \sin x = \frac{1}{2}\sin(2x)$$

(b)

$$\frac{1}{2}\sin(2x) = \frac{1}{2} \rightarrow x = k\pi + \frac{\pi}{4}, k \in \mathbb{Z}$$

2.1.4

Note that

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

and

$$\cos x = 2\cos^2\frac{x}{2} - 1 = 1 - 2\sin^2\frac{x}{2}$$

We have

$$\csc x + \cot x = \frac{1}{\sin x} + \frac{\cos x}{\sin x} = \frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)} = \cot\frac{x}{2}$$

Therefore

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right| = \ln\left|\frac{\cot(x_0/2)}{\cot(x/2)}\right| = \ln\left|\frac{\tan(x/2)}{\tan(x_0/2)}\right|$$

We then need to discuss the sign of the term $\tan(x/2)/\tan(x_0/2)$.

- If $x_0 \in (2k\pi, (2k+1)\pi), k \in \mathbb{Z}$

  It can be shown that $\forall t > 0, x(t) \in (x_0, (2k+1)\pi)$. Therefore $x_0/2, x/2 \in (k\pi, (k+1/2)\pi)$ and $\tan(x_0/2), \tan(x/2) > 0$.

- If $x_0 \in ((2k-1)\pi, 2k\pi), k \in \mathbb{Z}$

  It can be shown that $\forall t > 0, x(t) \in ((2k-1)\pi, x_0)$. Therefore $x_0/2, x/2 \in ((k-1/2)\pi, k\pi)$ and $\tan(x_0/2), \tan(x/2) < 0$.

We then have $\tan(x/2)/\tan(x_0/2) > 0$. Hence,

$$\begin{aligned}
&\tan\frac{x}{2} = \mathrm{e}^t \tan\frac{x_0}{2} \\
\rightarrow &x = 2\tan^{-1}\left(\mathrm{e}^t\tan\frac{x_0}{2}\right) + 2k\pi, \text{ where } x_0 \in ((2k-1)\pi, (2k+1)\pi)
\end{aligned}$$

It is worthy noticing that when $x_0 = k\pi, k \in \mathbb{Z}$, the system is already in the fix point. Therefore $x(t) = x_0 = k\pi$.

Especially, when $x_0 = \pi/4$, we have $k = 0$ and

$$\tan\frac{x_0}{2} = \frac{1}{\csc x_0 + \cot x_0} = \frac{1}{\sqrt{2}+1}$$

Then,

$$x = 2\tan^{-1}\left(\frac{\mathrm{e}^t}{1+\sqrt{2}}\right)$$

2.2 Fixed Points and Stability

2.2.1

- Fix points and stability

  $x = -2$, Stable; $x = 2$, Unstable.

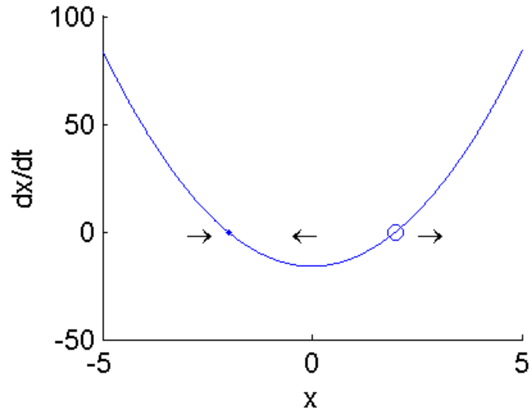

Figure 2: 2.2.1

- Analytical solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x^2 - 16$$
$$\frac{\mathrm{d}x}{4x^2 - 16} = \mathrm{d}t$$
$$\frac{1}{16}\left(\frac{1}{x-2} - \frac{1}{x+2}\right)\mathrm{d}x = \mathrm{d}t$$
$$\mathrm{d}\ln\left|\frac{x-2}{x+2}\right| = \mathrm{d}(16t)$$
$$\ln\left|\frac{x-2}{x+2}\right| - \ln\left|\frac{x_0-2}{x_0+2}\right| = 16t$$
$$\ln\left|\frac{x-2}{x+2}\frac{x_0+2}{x_0-2}\right| = 16t$$

  - If $x_0 > 2$, we have $x > 2$ and $(x-2)/(x+2), (x_0-2)/(x_0+2) > 0$;
  - If $-2 < x_0 < 2$, we have $-2 < x < 2$ and $(x-2)/(x+2), (x_0-2)/(x_0+2) < 0$;
  - If $x_0 < -2$, we have $x < -2$ and $(x-2)/(x+2), (x_0-2)/(x_0+2) > 0$.

Therefore, no matter what the initial value is, the term $(x-2)(x_0+2)/(x+2)(x_0-2)$ is positive. Therefore

$$\frac{x-2}{x+2}\frac{x_0+2}{x_0-2} = \mathrm{e}^{16t} \to x = 2\left(1 + \frac{x_0-2}{x_0+2}\mathrm{e}^{16t}\right)\Big/\left(1 - \frac{x_0-2}{x_0+2}\mathrm{e}^{16t}\right)$$

It is worth noticing that when $x_0 = -2$, the system is in the fix point. Therefore $x(t) = x_0 = -2, t > 0$. From now on, we will ignore the cases where the initial state is the fix point.

2.2.2

- Fix points and stability

  $x = -1$, Unstable; $x = 1$, Stable.

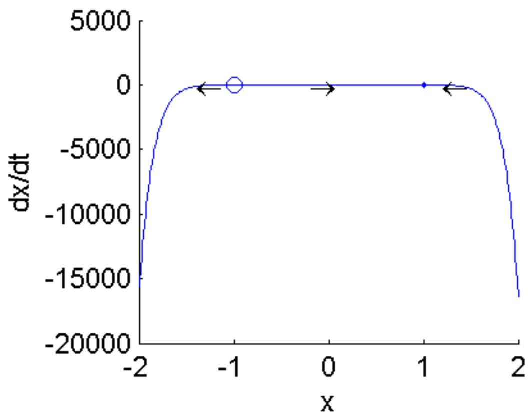

Figure 3: 2.2.2

- Analytical solution

  Note that the solutions of $1 - x^{14} = 0$ is $x = e^{in\pi/7}, n = 0, 1, \ldots, 13$. Therefore we have

$$
\begin{aligned}
&\frac{dx}{dt} = 1 - x^{14} \\
&\frac{dx}{\prod_{n=0}^{13}(\exp\{in\pi/7\} - x)} = dt \\
&\sum_{n=0}^{13} \frac{A_n}{\exp\{in\pi/7\} - x} dx = dt \\
&-\sum_{n=0}^{13} A_n d \ln \left| x - e^{(in\pi/7)} \right| = dt \\
&- d \prod_{n=0}^{13} \ln \left| x - e^{(in\pi/7)} \right|^{A_n} = dt \\
&\prod_{n=0}^{13} \ln \left| x_0 - e^{(in\pi/7)} \right|^{A_n} - \prod_{n=0}^{13} \ln \left| x - e^{(in\pi/7)} \right|^{A_n} = t
\end{aligned}
$$

where A_n are coefficients. The form is very complicated and it is not necessary to proceed.

2.2.3

- Fix points and stability

$x=-1$, Stable; $x=0$, Unstable; $x=1$, Stable.

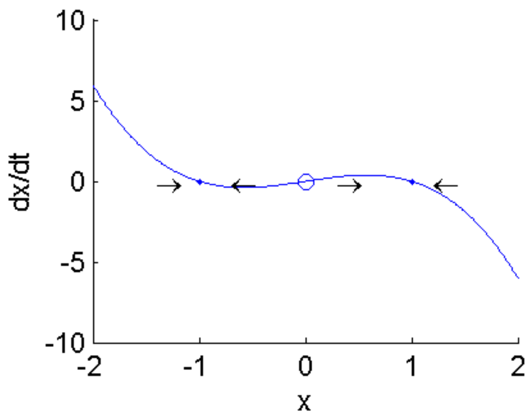

Figure 4: 2.2.3

- Analytical solution

$$
\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = x - x^3 \\
&\frac{\mathrm{d}x}{x - x^3} = \mathrm{d}t \\
&\left(\frac{1}{x} + \frac{1}{2}\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x}\right)\mathrm{d}x = \mathrm{d}t \\
&\mathrm{d}\ln|x| - \frac{1}{2}\mathrm{d}\ln|x-1| - \frac{1}{2}\mathrm{d}\ln|x+1| = \mathrm{d}t \\
&\mathrm{d}\ln\frac{|x|}{\sqrt{|x^2-1|}} = \mathrm{d}t \\
&\ln\frac{|x|}{\sqrt{|x^2-1|}} - \ln\frac{|x_0|}{\sqrt{|x_0^2-1|}} = t \\
&\frac{|x|}{\sqrt{|x^2-1|}} = \frac{|x_0|}{\sqrt{|x_0^2-1|}}\mathrm{e}^t \\
&\frac{x^2}{|x^2-1|} = \frac{x_0^2}{|x_0^2-1|}\mathrm{e}^{2t}
\end{aligned}
$$

  – If $|x_0| < 1$, we have $|x| < 1$. In addition, x and x_0 should have the same sign. So

$$\frac{x^2}{1-x^2} = \frac{x_0^2}{1-x_0^2}\mathrm{e}^{2t} \to x = \sqrt{\frac{\mathrm{e}^{2t}x_0^2/(1-x_0^2)}{1+\mathrm{e}^{2t}x_0^2/(1-x_0^2)}}\mathrm{sgn}x_0$$

  – If $|x_0| > 1$, we have $|x| > 1$. x and x_0 should have the same sign, too. So

$$\frac{x^2}{x^2-1} = \frac{x_0^2}{x_0^2-1}\mathrm{e}^{2t} \to x = \sqrt{\frac{\mathrm{e}^{2t}x_0^2/(x_0^2-1)}{\mathrm{e}^{2t}x_0^2/(x_0^2-1)-1}}\mathrm{sgn}x_0$$

2.2.4

- Fix points and stability
  $x = 2k\pi$, Unstable; $x = (2k+1)\pi$, Stable. $k \in \mathbb{Z}$

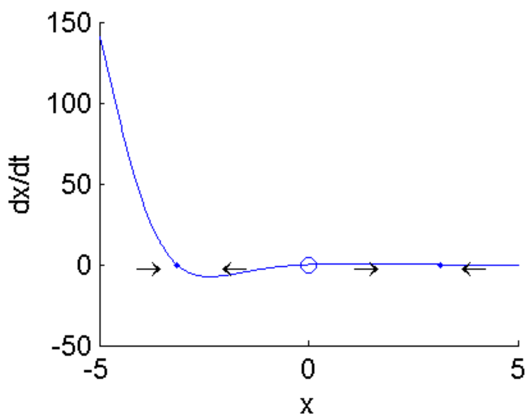

Figure 5: 2.2.4

- No analytical solution

2.2.5

- No fix points

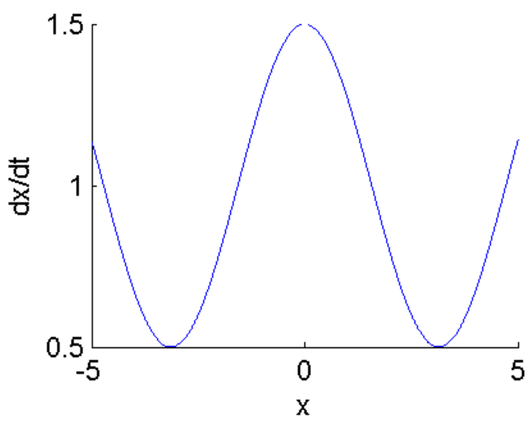

Figure 6: 2.2.5

- Analytical solution

Let $u = \tan(x/2)$, and we have $x = 2\arctan u + 2k\pi, k \in \mathbb{Z}$. Note that

$$1 + \frac{1}{2}\cos x = \frac{1}{2} + \cos^2\frac{x}{2} = \frac{1}{2} + \frac{1}{1+u^2}$$

and

$$\mathrm{d}x = \frac{2}{1+u^2}\mathrm{d}u$$

and we have

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \frac{1}{2}\cos x \\
&\frac{\mathrm{d}x}{1 + (1/2)\cos x} = \mathrm{d}t \\
&\frac{1}{1/2 + 1/(1+u^2)}\frac{2}{1+u^2}\mathrm{d}u = \mathrm{d}t \\
&\frac{4\mathrm{d}u}{u^2+3} = \mathrm{d}t \\
&\frac{4}{\sqrt{3}}\mathrm{d}\arctan\frac{u}{\sqrt{3}} = \mathrm{d}t \\
&\frac{4}{\sqrt{3}}\left(\arctan\frac{u}{\sqrt{3}} - \arctan\frac{u_0}{\sqrt{3}}\right) = t \\
&u = \sqrt{3}\tan\left(\frac{\sqrt{3}}{4}t + \arctan\frac{u_0}{\sqrt{3}}\right)
\end{aligned}$$

where $u_0 = \tan(x_0/2)$. Note that when converting u back to x, we are not able to determine the number of period (k) directly.

2.2.6

- Fix points and stability

  $x = 2k\pi + \pi/3$, Unstable; $x = 2k\pi - \pi/3$, Stable. $k \in \mathbb{Z}$

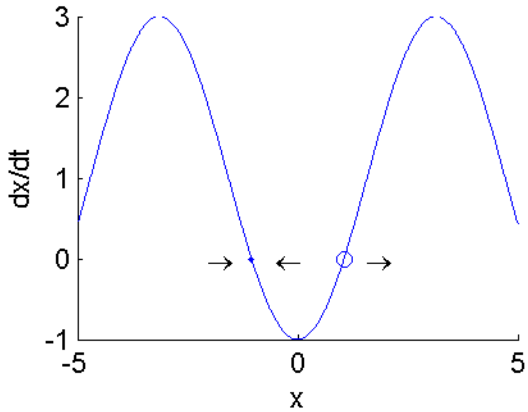

Figure 7: 2.2.6

- Analytical solution

  Similar to 2.2.5, define $u = \tan(x/2)$ and eventually we have

$$\frac{2}{3u^2 - 1}\mathrm{d}u = \mathrm{d}t$$

$$\frac{1}{\sqrt{3}}\mathrm{d}\ln\left|\frac{u - 1/\sqrt{3}}{u + 1/\sqrt{3}}\right| = \mathrm{d}t$$

$$\left|\frac{u - 1/\sqrt{3}}{u + 1/\sqrt{3}}\right| = \left|\frac{u_0 - 1/\sqrt{3}}{u_0 + 1/\sqrt{3}}\right| \mathrm{e}^{\sqrt{3}t}$$

  – If $x_0 \in (2k\pi - \pi/3, 2k\pi + \pi/3), k \in \mathbb{Z}$, we have $x \in (2k\pi - \pi/3, x_0)$ and $(u - 1/\sqrt{3})(u + 1/\sqrt{3}), (u_0 - 1/\sqrt{3})(u_0 + 1/\sqrt{3}) < 0$.
  – If $x_0 \in (2k\pi + \pi/3, 2k\pi + 5\pi/3), k \in \mathbb{Z}$, we have $x \in (x_0, 2k\pi + 5\pi/3)$ and $(u - 1/\sqrt{3})(u + 1/\sqrt{3}), (u_0 - 1/\sqrt{3})(u_0 + 1/\sqrt{3}) > 0$.

  So $(u - 1/\sqrt{3})(u + 1/\sqrt{3})$ and $(u_0 - 1/\sqrt{3})(u_0 + 1/\sqrt{3})$ have the same sign. Then we have

$$\frac{u - 1/\sqrt{3}}{u + 1/\sqrt{3}} = \frac{u_0 - 1/\sqrt{3}}{u_0 + 1/\sqrt{3}}\mathrm{e}^{\sqrt{3}t} \rightarrow u = \frac{1}{\sqrt{3}}\frac{\mathrm{e}^{\sqrt{3}t}(u_0 - 1/\sqrt{3})/(u_0 + 1/\sqrt{3}) + 1}{1 - \mathrm{e}^{\sqrt{3}t}(u_0 - 1/\sqrt{3})/(u_0 + 1/\sqrt{3})}$$

  where $u_0 = \tan(x_0/2)$. Note that when converting u back to x, we need to choose k carefully as the number of period may change.

2.2.7

- Fix points and stability

  Infinite fix points. The greatest one (the only positive one) is unstable. From high to low, the fix points are alternatively unstable and stable.

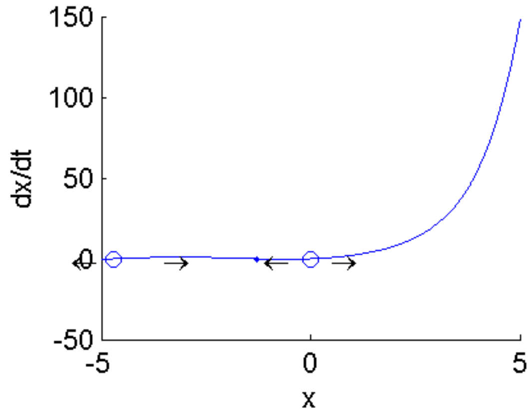

Figure 8: 2.2.7

- No analytical solution

2.2.11

We use the variation of constant method. First we have

$$\dot{Q} = -Q/RC \rightarrow Q = A\mathrm{e}^{-t/RC}$$

where A is a constant. Then we change A to a function of t, and we have

$$\mathrm{e}^{-t/RC}\dot{A} = V_0/R \rightarrow A = V_0 C\mathrm{e}^{t/RC} + A_0$$

where A_0 is a constant. Therefore, the expression of Q is

$$Q = V_0 C + A_0\mathrm{e}^{-t/RC}$$

Note that at time $t = 0$, we have $Q(0) = 0$. So $A_0 = -V_0 C$ and

$$Q = V_0 C\left(1 - \mathrm{e}^{-t/RC}\right)$$

2.3 Population Growth

2.3.2

Note that $k_1, a, k_{-1} > 0$. The fix points (and their stability) are $x = 0$ (unstable) and $x = k_1 a/k_{-1}$ (stable).

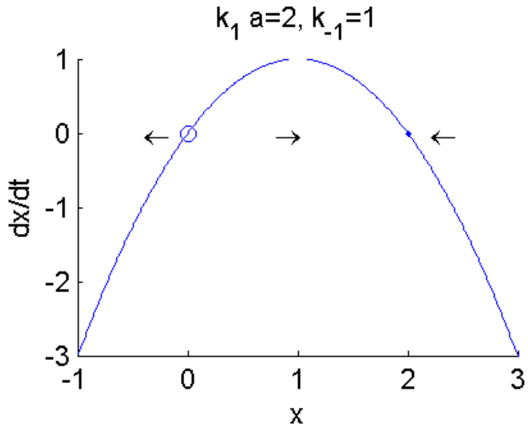

Figure 9: 2.3.2

When plotting $x(t)$, remind that the concentration of a chemical species should be non-negative. Therefore, it is wrong to choose a negative x_0.