Hints for excersize session 5

Solution strategy

We are faced with two nonlinear differential equations:

$$\begin{aligned}\dot{x} =& \quad f(x,y)\\ \dot{y} =& \quad g(x,y)\end{aligned}$$

First calculate the Jacobian, by taking the partial derivatives

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{x^*,y^*}$$

Then sketch the nullclines

This is the *line where either* $\dot{x}$ or $\dot{y}$ is equal to zero. For instance if $\dot{x} = x - y$ then the nullcline for $\dot{x}$ is the line $y = x$ which can be sketched. On the nullcline where $\dot{x}$ is equal to zero we only have a flow in the y-direction. To determine the direction of this, we must look at the equation for $\dot{y}$. If $\dot{y} = y^3$ then along the line $y = x$ the flow points up when $y > 0$ and it points down when $y < 0$.

Then find the fixed points

These can either be found by simply looking at the equations, or taking advantage of the fact that we have a fixed point where two nullclines intersect eachother.

Then characterize the fixed ponts and draw the phase portrait

Here we insert the values of our fixed points in the jacobian matrix, so we now have a matrix with four numbers. Use these to calculate τ and Δ and then use the characterisation scheme as seen in figure 1. Use the flow at the nullclines, your knowledge of the fixed points and common sense to draw the phase portrait.

° **On the Jacobian**

We consider a system which can be written as:

$$\begin{aligned}\dot{x} =& \quad f(x,y)\\ \dot{y} =& \quad g(x,y)\end{aligned}$$

At the fixed point we now have:

$$f(x^*,y^*) = 0 \quad g(x^*,y^*) = 0$$

We now make the substitution:

$$u = x - x^* \quad v = y - y^*$$

Now since x^* is just a constant we can write:

$$\begin{aligned}\dot{u} &= \dot{x}\\ &= f(u + x^*, v + y^*)\\ &= f(x^*,y^*) + u\ \frac{df}{dx}\Big|_{x^*,y^*} + v\ \frac{df}{dy}\Big|_{x^*,y^*}\\ &= u\ \frac{df}{dx}\Big|_{x^*,y^*} + v\ \frac{df}{dy}\Big|_{x^*,y^*}\end{aligned}$$

Here we have neglected terms of the order v^2, u^2 and higher terms. This is because that v and u are very small close to the fixed point. If we do the same for $\dot{v}$ we can write it in matrix notation:

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

The matrix evaluated at a specific fixed point is known as the Jacobian of that fixed point. Note that we use the jacobian to *characterize* the fixed points, to see how small perturbations evovle. We don't use it to *solve* the actual system.

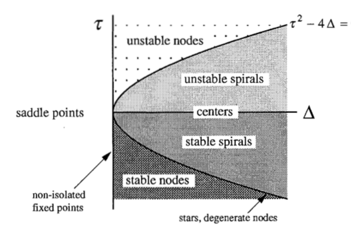

Figure 1: This figure is extremely useful when characterizing fixed points

° **Hint for 6.3.8**

Show that we have a fixed point at:

$$x = a\frac{-m_1 \pm \sqrt{m_1 m_2}}{m_2 - m_1}$$

Insert this in the jacobian, and consider the value of τ and Δ for $m_1 > m_2$ and $m_1 < m_2$