Svar i Eksamensopgaver til Dynamiske Systemer og Kaos, 1. april 2009.

I.

1. $x^* = \frac{1}{2}(1 \pm \sqrt{1 - \frac{4}{r}}$
2. $1 - \frac{4}{\geq} 0 \Rightarrow r \geq 4, r < 0$. Fixed points in $]-\infty; 0[$ and $[4; \infty[$.
3. Insert fixed points: $f'(x) = 2rx - r = \pm\sqrt{r^2 - 4r}$. The '+' branch unstable, '-' branch stable.
4. Saddle node bifurcation at $(r, x^*) = (4, \frac{1}{2})$. Upper branch unstable, lower branch stable. As $r \to -\infty$ there are two asymptotes, $x^* = 1$, upper unstable branch, and $x^* = 0$ lower stable branch. As $r \to 0^-$ the upper branch diverges to $+\infty$ and the lower branch to $-\infty$
5. Saddle node.

II.

6. $\dot{x}_1 = x_2 \quad , \quad \dot{x}_2 = -g\sin x_1 - bx_2$
7. The height above the equilibrium state of the pendulum is $1 - \cos x_1$ and therefore the potential energy of a pendulum is $g(1 - \cos x_1)$, and the kinetic energy is given by the squared velocity $\frac{1}{2}x_2^2$.
8. Lyapunov function: $E(x_1, x_2) > 0$ when $(x_1, x_2) \neq (0, 0)$

   $E(0, 0) = 0$

   $\dot{E} = g\sin x_1 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2 = g\sin x_1 \cdot x_2 - gx_2 \cdot \sin x_1 - bx_2^2 = bx_2^2 < 0$
9. $(0, 0), (\pi, 0)$ are fixed points. Eigenvalues: $(0, 0) : \lambda = (-b \pm \sqrt{b^2 - 4g})/2 \ , \ (\pi, 0) : \lambda = (-b \pm \sqrt{b^2 + 4g})/2$
10. The $(0, 0)$ fixed point with $g = 1$: For $b \geq 2$ two negative real eigenvalues. For $b < 2$ two complex eigenvalues where the pendulum oscillates back and forth before it comes to a rest.
11. $(0, 0)$ becomes unstable at $g = 0$ (gets a positive eigenvalue). The pendulum begins to move upwards.

III.

12. $\dot{r} = r(1 - r) = 0 \Rightarrow r^* = 0(unstable), r^* = 1(stable)$
13. $\frac{1}{r} + \frac{1}{1-r} = \frac{1-r+r}{r(1-r)}$.
14. $\int_{r_0}^{r_1}(\frac{1}{r} + \frac{1}{1-r})dr = \int_0^{2\pi} dt \Rightarrow [\ln\frac{1-r}{r}]_{r_0}^{r_1} = -2\pi \Rightarrow (\frac{1}{r_1} - 1)/(\frac{1}{r_0} - 1) = e^{-2\pi} \Rightarrow r_1 = \frac{1}{1+(\frac{1}{r_0}-1)e^{-2\pi}}$
15. Graph of the function $P(r_0)$ is above the diagonal close to zero. It crosses the diagonal at $r_0 = r^* = 1$ with a small positive slope.
16. $P'(r_0) = \frac{e^{-2\pi}/r_0^2}{(1+(\frac{1}{r_0}-1)e^{-2\pi})^2}$ In the fixed point $r_0 = r^* = 1$ we find $P'(1) = e^{-2\pi} \sim 0.002$. As $|P'(1)| < 1$ the fixed point is stable.