I.

1. Fixed points: $(0,0), (-1,-1)$.
2. Eigenvalues for $(0,0)$: $\lambda = \pm 1$, a saddle-point.

   Eigenvalues for $(-1,-1)$: $\lambda = -1, -3$, a stable node.
3. Eigenvectors: For $(0,0)$: $(1,1)(\lambda = 1), (1,-1)(\lambda = -1)$. For $(-1,-1)$: $(1,0)(\lambda = -1), (0,1)(\lambda = -3)$. Stable manifold for $(0,0)$ (with eigenvalue $\lambda = -1$) defines boundary for basin of attraction for stable fixed point $(-1,-1)$.

II.

4. $(0,0)$ is a fixed point. Another fixed point must satisfy $x^* = 1$ and $x^* = -2y^{*2}$ but that is not consistent. Therefore $(0,0)$ is the only fixed point.
5. We assume $V(x,y) = x^2 + ay^2$. Then $\dot{V}(x,y) = 2x\dot{x} + 2ay\dot{y} = -x^2 - x^2y^2 + (a-2)xy^2$. If we choose $a = 2$ then $\dot{V}(x,y) < 0$ and $V(0,0) = 0$.

III.

6. We identify $h(x,\dot{x}) = \frac{1}{3}\dot{x}^3 - \dot{x}$. With $x = r(T)cos(\theta), \dot{x} = -r(T)sin(\theta)$ we get $r' =< -\frac{1}{3}r^3 sin^4(\theta) + rsin^2(\theta) >= -\frac{1}{8}r^3 + \frac{1}{2}r$.
7. $r\phi' =< -\frac{1}{3}r^3 sin^3(\theta)cos(\theta) + rsin(\theta)cos(\theta) >= 0$.
8. $\frac{dr}{dT} = -\frac{1}{8}r^3 + \frac{1}{2}r \Rightarrow dT = \frac{8}{r(4-r^2)}dr = (\frac{2}{r} + \frac{2r}{4-r^2})dr \Rightarrow \int dT = \int(\frac{2}{r} + \frac{2r}{4-r^2})dr \Rightarrow ln\frac{r^2}{4-r^2} = T + k$. Since $r(0)^2 = x(0)^2 + \dot{x}(0)^2 = 2$ we obtain $k = 0$. That gives the solution $r(T) = \frac{2}{\sqrt{1+e^{-T}}} \Rightarrow r(t) = \frac{2}{\sqrt{1+e^{-ct}}}$.

IV.

9. $\frac{\partial H}{\partial \phi_n} = -(\phi_{n+1} - \phi_n) + (\phi_n - \phi_{n-1}) + a\phi_n(\phi_n^2 - 1) = 0 \Rightarrow \phi_{n+1} + \phi_{n-1} - 2\phi_n = a\phi_n(\phi_n^2 - 1)$.
10. $x_{n+1} = \phi_{n+1} = 2x_n + ax_n(x_n^2 - 1) - \phi_{n-1} = 2x_n + ax_n(x_n^2 - 1) - y_n$

    $y_{n+1} = \phi_n = x_n$.
11. Determinant of Jacobian $= 1$.
12. $(0,0), (1,1), (-1,-1)$ are fixed points for T.
13. Eigenvalues for $(0,0) : \lambda = \frac{2-a\pm\sqrt{a^2-4a}}{2}$. For $a < 4$ they are complex eigenvalues on the unit circle.

    Eigenvalues for $(1,1), (-1,-1) : \lambda = 1 + a \pm \sqrt{2a + a^2}$.

    The "+" eigenvalue is for $a > 0$ always greater that 1, i.e. an unstable eigendirection.

    The "-" eigenvalue is for $a > 0$ always smaller that 1, i.e. a stable eigendirection.
14. For the fixed point $(0,0)$ the complex eigenvalues $\lambda = \frac{2-a\pm\sqrt{a^2-4a}}{2}$ "collide" for $a = 4$ at the value $\lambda = -1$. Therefore a period doubling will take place.