Eksamen består af 3 opgaver. Spørgsmål 1-15 indgår med lige vægt i bedømmelsen.
Svarene kan skrives på dansk og/eller engelsk. Blyant er tilladt.
Eksamensresultat og løsninger opslås på hjemmesiden.

I.

Consider the differential equation

$$\dot{x} = f(x, r) = x^2 - rx - r \tag{1}$$

1. Find the fixed points for (1) as a function of r.
2. Identify two bifurcation points and find the critical values of the parameter, $r_{c1} < 0$ and $r_{c2} \geq 0$.
3. Determine the stability of the fixed points.
4. What type of bifurcation takes place at r_{c1} and at r_{c2}.
5. Determine the asymptotic behavior of the fixed points as a function of r for large, positive values of r (Hint: Use Taylor expansion).
6. Sketch the bifurcation diagram, i.e. fixed points x^* as a function of r (full line: stable fixed points, dashed line: unstable fixed points).

II.

Consider the discrete mapping

$$x_{n+1} = f(x_n) = \begin{cases} 4x_n & \text{if} \quad x_n \leq 1/4 \\ 2 - 4x_n & \text{if} \quad 1/4 \leq x_n \leq 3/4 \\ 4x_n - 4 & \text{if} \quad 3/4 \leq x_n \end{cases} \tag{2}$$

7. Determine the fixed points of (2) and their stability.

   A cycle-point x_n^* can be associated with the letter L if $x_n^* \in]-\infty; 1/4]$, with the letter M if $x_n^* \in [1/4; 3/4]$ and with the letter R if $x_n^* \in [1/4; \infty[$.
8. Find the orbit points of the two-cycle, where $x_n^* = f^2(x_n^*)$, of the mapping (2) (and different from the fixed point) which in terms of these symbols can be described as $LMLMLM$.....
9. Determine the stability of the two-cycle.
10. The mapping (2) has a three-cycle, where $x_n^* = f^3(x_n^*)$ (and different from the fixed point) which in terms of the symbols can be described as $LLMLLMLLM$..... Determine the orbit points of this three-cycle.
11. For which interval of initial points x_0 will the sequences $x_n \to +\infty$ for large n.

III.

For the second-order differential equation

$$\ddot{x} + x + \epsilon(x^4 - a)\dot{x} \;=\; 0 \;,\; a > 0 \tag{3}$$

12. Use averaging theory (or equivalently two timing) to derive equations for r' and ϕ'.
13. Find two fixed points for $r(T)$.
14. Determine the stability of these fixed points.
15. Difficult: Set $a = 2$. Write the solution for $r(T)$ with the boundary condition $r(0) = 1$. (Hint: Use the 'splitting': $\frac{16}{r(16-r^4)} = \frac{c}{r} + \frac{d}{16-r^4}$ and determine the coefficients c and d). Describe the flow when $T \to \infty$.