Eksamenssvar til opgaver til Fysik 222, 10. juni 2002.

I.

1. $A(b,d) = \frac{1}{\sqrt{bd}}$, $B(b,d) = \frac{1}{b}$.
2. $(x^*, y^*) = (0,0), (\sqrt{a}, a), (-\sqrt{a}, a)$.
3. $(0,0)$ is stable for $a < 0$ and unstable for $a \geq 0$.
   $(\sqrt{a}, a)$ and $(-\sqrt{a}, a)$ are stable for $a \geq 0$ (and do not exists for $a < 0$).
4. A stable fixed point $(0,0)$ splits into two stable fixed point and becomes unstable itslef: A pitchfork bifurcation takes place at $a = 0$.
5. For $a > 1/8$, the fixed points $(\sqrt{a}, a)$ and $(-\sqrt{a}, a)$ assume complex eigenvalues so the trajectories will spiral around them, but they are still stable.

II.

6. When $x \to y$, $y \to x$, $t \to -t$ then the system stays invariant only for $a = -1$.
7. For $a = -1$ the fixed point $(0,0)$ is a linear center. It is also reversible (Strogatz p. 167), therefore it is a non-linear center and has invariant curves around.
8. For $a = -1$ the line of fixed points are the hyperpolas $(x, 1/x)$.
9. For $a = -1$ one can show for $r^2 = x^2 + y^2$ that $\dot{r} = 0$, so the trajectories around $(0,0)$ are circles. The line of fixed points $(x, 1/x)$ are stable. The initial point $(2,0)$ starts out along the vector $(\dot{x}, \dot{y}) = (0, -2)$ and will move on the curve $x^2 + y^2 = 4$. It ends in a point that is one of the solutions to $x^2 + \frac{1}{x^2} = 4$ (the one with negative x and closest to $y = 0$): $x^* = \lim_{t\to\infty} x(t) = -\sqrt{2 - \sqrt{3}}$.

III.

10. $x^* = 0$, $x^* = \frac{3}{5}$. $f'(0) = 3$, i.e. unstable. $f'(3/5) = -\frac{3}{2}$, i.e. unstable.
11. $x_1^* = \frac{3}{11} \to x_2^* = \frac{9}{11} \to x_1^*$.
    $f^{2\prime}(x_1^*) = f'(x_1^*) \cdot f'(x_2^*) = -\frac{9}{2}$, i.e. the two-cycle is unstable.
12. The possible three cycles: $LLRLLR....$ and $LRRLRR.....$
13. One three cycle: $LLR....$: $x_1^* = \frac{3}{29} \to x_2^* = \frac{9}{29} \to x_3^* = \frac{27}{29} \to x_1^*$
    The other three cycle: $LRR....$: $x_1^* = \frac{3}{23} \to x_2^* = \frac{9}{23} \to x_3^* = \frac{21}{23} \to x_1^*$
14. The derivative in a cycle point is either 3 or -$\frac{3}{2}$. Therefore the absolute values of the product of eigenvalues are always larger than one, and all cycles are therefore unstable.