Eksamensopgaver til Fysik 222, 6. juni 2001.
Varighed 3 timer. Hjælpemidler tilladt. Eksamen består af 3 opgaver.
Spørgsmål 1-14 indgår med lige vægt i bedømmelsen.
Eksamensresultat og løsninger opslås på hjemmesiden.

I.

For the differential equation

$$\dot{x} = f(x) = r \ \sin x \ - \ \sin \ 2x \tag{1}$$

1. $\dot{x} = 0 \Rightarrow x^* = 0$ or $x^* = \pi$ are always fixed points for any value of r. For $x \to -x$ all terms change sign.
2. $f'(x^*) = r\cos x^* - 2\cos\ 2x^*$. $f'(0) = r - 2 = 0 \Rightarrow r_c = 2$. $f'(\pi) = -r - 2 \Rightarrow \tilde{r} = -2$.
3. $\dot{x} = 0 \Rightarrow r \sin x^* - \sin\ 2x^* = r \sin x^* - 2\sin\ x^* \cos\ x^* = \sin x^*(r - 2\cos\ x^*) = 0 \Rightarrow \cos\ x^* = \frac{r}{2}$.
4. $f'(x^*) = r\cos x^* - 2\cos\ 2x^* = r\cos x^* - 2(2\cos^2 x^* - 1) = 2 - \frac{r^2}{2}$. The new fixed points exist only in the interval $r \in [-2; 2]$ and since $f'(x^*) \geq 0$ in this interval they are always unstable.
5. $r \simeq 2 \Rightarrow x^* \simeq 1 \Rightarrow \cos x^* \simeq 1 - \frac{x^{*2}}{2} = \frac{r}{2} \Rightarrow x^* = \pm\sqrt{2-r}$. $(r \leq 2)$.
6. Pichfork bifurcation, sub-critical.
7. $x^* \simeq \pi \Rightarrow \cos\ (x^* - \pi) \simeq -1 + \frac{(x^*-\pi)^2}{2} = \frac{r}{2} \Rightarrow (x^* - \pi) = \pm\sqrt{r+2}$ $(r \geq -2)$.
8. Fixed point $x^* = 0$ unstable for $r \geq 2$ (dashed line) and stable for $r < 2$ (full line). Fixed point $x^* = \pi$ unstable for $r \leq -2$ (dashed line) and stable for $r > -2$ (full line). Fixed points $\cos\ x^* = \frac{r}{2}$ in $r \in [-2; 2]$ are always unstable (dashed). Around r_c bifurcated fixed points vary as a squareroot. Around $\tilde{r}$ bifurcated fixed points also vary as a squareroot. System is symmetric around $x^* = 0$ and is periodic in x^* with period 2π. Fixed points go through $(0, \frac{\pi}{2})$ (and thus $(0, -\frac{\pi}{2})$).

II.

9. $\dot{x} = 0 \Rightarrow -4y^2 = \sqrt{x^2+y^2} \Rightarrow y^* = 0 \Rightarrow x^* = 0$. Therefore $(x^*, y^*) = (0, 0)$ is the only fixed point.
10. $\dot{V}(x,y) = 2x\dot{x} + 2ay\dot{y} = (-8+2a)x^2y^2 - 2x^2\sqrt{x^2+y^2} - 2ay^2\sqrt{x^2+y^2}$. First term negative: $a < 4$. Then also last term (-$2ay^2\sqrt{x^2+y^2}$) is negative if $a > 0$. We also demand $V(x,y) > 0$ if $(x^*, y^*) \neq (0,0)$. Therefore for $0 < a < 4$, $V(x,y) > 0$ and $\dot{V}(x,y) < 0$ if $(x^*, y^*) \neq (0,0)$. Therefore: $(x^*, y^*) = (0,0)$ is a stable fixed point.

III.

For the logistic map

$$x_{n+1} = rx_n(1 - x_n) \tag{2}$$

11. $a\tilde{x}_{n+1} + b = ra\tilde{x}_n + rb - r(a\tilde{x}_n + b)^2 \Rightarrow \tilde{x}_{n+1} = \frac{-b+rb-rb^2}{a} + r(1-2b)\tilde{x}_n - ra\tilde{x}_n^2 \Rightarrow a = \frac{1}{r}$, $b = \frac{1}{2}$, $R = \frac{1}{4}r^2 - \frac{1}{2}r$.
12. $x_0 = R - (R - x_0^2)^2 = R - R^2 - x_0^4 + 2Rx_0^2$. In the superstable case (see Strogatz ex. 10.7.1, page 380), $x_0 = 0 \Rightarrow R - R^2 = 0 \Rightarrow R = 1$.
13. $f_R^2{}'(x_0) = f_R'(x_0) \cdot f_R'(f(x_0)) = -2x_0 \cdot -2(R - x_0)^2 = 4x_0R - 4x_0^3$
14. At period doubling bifurcation: $f_R^2{}'(x_0) = -1 \Rightarrow x_0^3 - x_0R - \frac{1}{4} = 0$.