Eksamensopgaver til Fysik 222, 30. januar 2002.
Varighed 3 timer. Hjælpemidler tilladt. Eksamen består af 3 opgaver.
Spørgsmål 1-14 indgår med lige vægt i bedømmelsen.
Eksamensresultat og løsninger opslås på hjemmesiden.

I.

Consider the differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(\sqrt{x^2+y^2}-\mu)y - x \end{aligned} \tag{1}$$

1. Show that $(x, y) = (0, 0)$ is a fixed point of (1) for all values of μ. Find the eigenvalues for this trivial fixed point and show that the real parts vanish when $\mu = 0$.
2. Which kind of solutions are expected to bifurcate at $\mu = 0$?
3. Change to polar coordinates (r, θ). Construct the resulting dynamical system $\dot{r} = h_1(r, \theta)$, $\dot{\theta} = h_2(r, \theta)$.
4. Show that in the case $\mu \ll 1$ and $r \ll 1$, the equation for $\dot{\theta}$ can be integrated to $\theta(t) = -t + \theta_0$ (plus terms of order μ and of order r).
5. Apply averaging theory to derive an effective equation $\dot{r} = h(r)$. Use this equation to find the fixed points r^* of r.
6. Investigate the stability of the trivial fixed point $r^* = 0$ as well as the nontrivial fixed point in a neighborhood of $\mu = 0$.
7. Plot the bifurcation diagram (i.e. (μ, r^*)).
8. Is this bifurcation around $\mu \simeq 0$ a *generic* Hopf bifurcation ?

II.

Consider the discrete mapping

$$x_{n+1} = f_r(x_n) = rx_n + x_n^3 \tag{2}$$

9. Determine the fixed points x_n^* of (2).
10. Show that $x_n^* = 0$ is a stable fixed point in an interval $[r_-, r_+]$ of the parameter.
11. Draw the bifurcation diagram (r, x_n^*) (full line: stable fixed points, dotted line: unstable fixed points).
12. What kind of bifurcation takes place for $x_n^* = 0$ around $r \simeq r_+$?
13. What kind of bifurcation takes place for $x_n^* = 0$ around $r \simeq r_-$?
14. Show that a two-cycle with elements $x_\pm^c$ exists for $r = r_- - \epsilon$, where $\epsilon > 0$ is a small parameter. Express $x_\pm^c$ to lowest order in ϵ. (Hint: Start with the equation for the two-cycle elements, $x^c = f_r^2(x^c)$, and keep terms up to order ϵ and to order $(x^c)^3$.)