Eksamensopgaver til Fysik 222, 6. juni 2001.
Varighed 3 timer. Hjælpemidler tilladt. Eksamen består af 3 opgaver.
Spørgsmål 1-14 indgår med lige vægt i bedømmelsen.
Eksamensresultat og løsninger opslås på hjemmesiden.

I.

Consider the differential equation

$$\dot{x} = r \sin x - \sin 2x, \qquad -\pi < x \leq \pi \tag{1}$$

1. Show that $x = 0$ and $x = \pi$ are fixed points for any value of the parameter r. Show that (1) is invariant under the transformation $x \to -x$.
2. Show that both fixed points undergo a bifurcation as r is varied. Find the bifurcation points $r_c > 0$ and $\tilde{r} < 0$.
3. Show that the bifurcations give rise to fixed points that satisfy
$$\cos x^* = \frac{r}{2} \tag{2}$$
4. Determine the stability of these fixed points using relation (2).
5. Close to the bifurcation point r_c (i.e $x^* \simeq 0$), Taylor expand relation (2) to second order and derive an approximation for the fixed points as a function of r below the bifurcation point ($r \leq r_c$).
6. What type of bifurcation takes place at r_c. Is it super- or sub-critical ?
7. A similar bifurcation takes place at $r = \tilde{r}$. Close to the bifurcation point $\tilde{r}$ (i.e $x^* \simeq \pi$), Taylor expand relation (2) to second order and derive an approximation for the fixed points as a function of r for $r \geq \tilde{r}$.
8. Sketch the bifurcation diagram for all values of r and $x^* \in]-\pi; \pi]$ (full line: stable fixed points, dashed line: unstable fixed points).

II.

For the system of differential equations

$$\begin{aligned} \dot{x} &= -4xy^2 - x\sqrt{x^2+y^2} \\ \dot{y} &= x^2y - y\sqrt{x^2+y^2} \end{aligned} \tag{3}$$

9. Show that the system has one fixed point.
10. Use the Liapunov function $V(x,y) = x^2 + ay^2$ for a suitable value of a to determine the stability of the fixed point (hint: determine an interval in the parameter a).

III.

Consider the logistic map

$$x_{n+1} = rx_n(1 - x_n) \tag{4}$$

11. Show that by the transformation $x_n = a\tilde{x}_n + b$, the logistic equation can be transformed to $\tilde{x}_{n+1} = R - \tilde{x}_n^2$. Determine a, b and R as a function of r for this transformation.
12. Now consider the map $f_R(x_n) = x_{n+1} = R - x_n^2$. Let x_0 be the starting point for the iteration. Write an equation (of 4'th order) in x_0 for the period two cycle of the map.
13. Write the derivative of the second iterate: $f_R^{2\,\prime}(x_0)$.
14. What is the condition for the period two cycle to undergo a period doubling bifurcation ?