Løsninger til Fysik 222, 20. juni 2005.

I.

1. $(x^*, y^*) = (\frac{1}{\mu}, \mu)$.

2. $\lambda(\mu) = \frac{1-\mu^2 \pm \sqrt{1-6\mu^2+\mu^4}}{2}$

3. When the expression under the square-root changes sign, the behavior of the fixed point changes, $\mu^2 = 3 + \sqrt{8}$.

4. At $\mu^2 = 3 - \sqrt{8}$ where the fixed point changes from being an unstable spiral to an unstable node.

5. The real part changes sign at $\mu = \pm 1$ which is where the Hopf bifurcation takes place.

II.

6. $\dot{x} = r \ \tan x \ - \ \sin 2x = \sin x(\frac{r}{\cos x} - 2\cos x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0$ or $x = \pi$.

7. $f'(x, r) = \frac{r}{\cos^2 x} - 2\cos 2x$. For both $x = 0$ and $x = \pi$ $f'(x, r) = r - 2$. Bifurcations takes place at $r_c = 2$.

8. $f(x^*, r) = 0 \Rightarrow \cos^2 x^* = \frac{r}{2}, \quad 0 \leq r \leq 2$.

9. $f'(x^*, r) = \frac{r}{\cos^2 x^*} - 4\cos^2 x + 2 = 4 - 2r$. Below the bifurcation point $r_c = 2$ (where $\cos x^* = \pm 1$) $x = 0$ and $x = \pi$ splits into two unstable fixed points and become themselves stable.

10. $cosx^* \simeq 1 - \frac{x^{*2}}{2} \Rightarrow (1 - \frac{x^{*2}}{2})^2 = \frac{r}{2} \Rightarrow 1 - x^{*2} \simeq \frac{r}{2} \Rightarrow x^* = \pm\sqrt{1 - \frac{r}{2}}$

    Alternatively: $\cos^2 x^* = \frac{r}{2} \Rightarrow \cos x^* = \sqrt{\frac{r}{2}} \Rightarrow 1 - \frac{x^2}{2} \simeq \sqrt{\frac{r}{2}} \Rightarrow x^* = \pm\sqrt{2(1 - \sqrt{\frac{r}{2}})}$

11. They are both sub-critical.

12. $x = 0$ is unstable for $r > 2$ and bifurcates into two unstable fixed points for $r \leq 2$ while it itself becomes stable.

    $x = \pi$ is unstable for $r > 2$ and bifurcates into two unstable fixed points for $r \leq 2$ while it itself becomes stable. One of the unstable fixed points is just below $x = \pi$, the other is just above $x = -\pi$

III.

13. $x^* = \pm\sqrt{r}$. Stability: $f'(x^*) = 1 - 2x^* = 1 \mp 2\sqrt{r}$. $+\sqrt{r}$ is stable for $0 \leq r \leq 1$. $-\sqrt{r}$ is unstable for all r.

14. $f'(x) = 1 - 2x = 0 \Rightarrow x_s = \frac{1}{2}$

15. $f'(x^* = +\sqrt{r}) = 1 - 2\sqrt{r} = -1 \Rightarrow r = 1$

16. $x_s = \frac{1}{2}$ must be part of a super-stable two cycle (otherwise the derivative cannot be zero). This two-cycle must therefore fulfill ($\rightarrow$ means 'mapped') $\frac{1}{2} \rightarrow \frac{1}{4} + r \rightarrow \frac{1}{4} + r + r - (r^2 + \frac{r}{2} + \frac{1}{16}) = \frac{1}{2} \Rightarrow 16r^2 + 5 - 24r = 0$. This has solutions $r = \frac{5}{4}$ and $r = \frac{1}{4}$. The first value corresponds to the two cycle $\frac{1}{2} \rightarrow \frac{3}{2} \rightarrow \frac{1}{2}$