Svar: Eksamensopgaver til Fysik 222, 10. juni 2004.

I.

1. Fixed points and limit cycles.
2. $\alpha = \frac{c}{a}, \beta = a, B = \frac{b}{a}$
3. $\dot{K} = \dot{x} + \dot{y} - B\frac{\dot{x}}{x} - \frac{\dot{y}}{y} = 0$, therefore K is conserved in time.

   (note: for $x > 0 : \frac{d}{dt}\ln|x| = \frac{d}{dt}\ln x = \frac{\dot{x}}{x}$; for $x < 0 : \frac{d}{dt}\ln|x| = \frac{d}{dt}\ln(-x) = \frac{\dot{x}(-1)}{-x} = \frac{\dot{x}}{x}$).
4. Fixed points: $(0,0)$ and $(B,1)$.

   Eigenvalues for $(0,0)$: $\lambda = 1, -B$, a saddle point, unstable.

   Eigenvalues for $(B,1)$: $\lambda = \pm i\sqrt{B}$, a linear center, marginally stable.
5. Eigenvectors: For $(0,0)$: $(1,0)(\lambda = 1)$, (unstable direction); $(0,1)$(stable direction)$(\lambda = -B)$.

   Nullclines: $x = 0$, $y = 1$, $y = 0$, $x = B$.

   Invariant curves around fixed point $(B,1)$ in 1'st quadrant. Flow goes to $-\infty$ along x-axis.

II.

6. $r\dot{r} = x\dot{x} + y\dot{y} = -xy - \epsilon(x^2-1)y^2 + xy = -\epsilon(x^2-1)y^2 \Rightarrow \dot{r} = (-\epsilon(x^2-1)y^2)/r$.
7. As $r \geq 0$ and trapping region defined by $\dot{r} < 0$ then $\dot{r} < 0 \Rightarrow -\epsilon(x^2-1)y^2 = \epsilon y^4 + \epsilon(1-r^2)y^2 < 0$. As ϵy^4 is always negative ($\epsilon < 0$) and y^2 always positive then $1 - r^2 > 0 \Rightarrow r^2 < 1 \Rightarrow r < 1$. With initial conditions inside any circle with radius $r < 1$, then the orbit will tend to the origin.
8. As $r < 1$ we have $r_m = 1$.

III.

9. $x^* = 1 - \mu x^{*2} \Rightarrow x^* = \frac{-1 \pm \sqrt{1+4\mu}}{2\mu}$.

   Stability: $f'(x^*) = -2\mu x^* = 1 \mp \sqrt{1+4\mu}$. The '+' solution is always unstable. The '-' solution becomes unstable when $f'(x^*) = -1 \Rightarrow +1 - \sqrt{1+4\mu} = -1 \Rightarrow \mu_1 = \frac{3}{4}$.
10. For a two-cycle: $x_+ = 1 - \mu x_-^2, x_- = 1 - \mu x_+^2 \Rightarrow x_+ - x_- = \mu(x_+ + x_-)(x_+ - x_-) \Rightarrow x_+ + x_- = 1/\mu$.
11. $x_- = 1 - \mu x_+^2 \Rightarrow x_+ + 1 - \mu x_+^2 = 1/\mu \Rightarrow x_\pm = \frac{1 \pm \sqrt{4\mu - 3}}{2\mu}$.
12. $f^{2\prime}(x_+) = f'(x_-) \cdot f'(x_+) = (-2\mu x_-) \cdot (-2\mu x_+) = 4 - 4\mu = 0 \Rightarrow \mu_s = 1$
13. $f(\frac{6}{7}) = 1 - \frac{7}{4}\frac{36}{49} = -\frac{2}{7}$

    $f(-\frac{2}{7}) = 1 - \frac{7}{4}\frac{4}{49} = \frac{6}{7}$

    Yes, it is a two-cycle. For $\mu = \frac{7}{4}$, $f^{2\prime}(\frac{6}{7}) = -3$ so it is unstable.
14. $f^{2\prime}(x_+) = 4 - 4\mu = -1 \Rightarrow \mu_2 = \frac{5}{4}$.