Eksamensopgaver til Dynamiske Systemer og Kaos, 1. april 2009.
Varighed 3 timer. Hjælpemidler (incl. lommeregner og computer) er tilladt.
Eksamen består af 3 opgaver. Spørgsmål 1-16 indgår med lige vægt i bedømmelsen.
Svarene kan skrives på dansk og/eller engelsk. Blyant er tilladt.
Eksamensresultat og løsninger opslås på hjemmesiden.

I.

Consider the differential equation

$$\dot{x} = rx^2 - rx + 1 \tag{1}$$

1. Determine the two intervals $]-\infty; r_1[$ and $[r_2; \infty[$ for which there are fixed point for equation (1).
2. Find the fixed points of (1) as a function of r in these intervals.
3. Determine the stability of the fixed points.
4. Sketch the bifurcation diagram, ie. fixed points as a function of r (full line: stable, dashed line: unstable).
5. What kind of bifurcation takes place at r_2.

II.

Consider a pendulum equation

$$\ddot{\theta} + a\sin\theta + b\dot{\theta} = 0 \tag{2}$$

6. Write it as a two-dimensional dynamical system with $x_1 = \theta$ and $x_2 = \dot{\theta}$.
7. Let $a > 0$ and show that the total (potential plus kinetic) energy of the system is $E = a(1 - \cos x_1) + \frac{1}{2}x_2^2$
8. Use this as a Lyapunov function to argue that $(0, 0)$ is a stable fixed point.
9. Find the fixed points of the two-dimensional system with $x_1 \in [0; 2\pi]$. Calculate the eigenvalues for the fixed points as a function of a and b.
10. Let now $a = 1$. Describe what happens for the fixed points at $b = 2$.
11. Consider the "unphysical" case where $a \leq 0$. For which value of a does the fixed point $(0, 0)$ become unstable. If $a < 0$ what would it mean for the pendulum.

III.

Consider the system of differential equations

$$\dot{r} = r(1 - r) \quad , \quad \dot{\theta} = 1 \tag{3}$$

12. Find the fixed points for the radius r. Determine their stability.
13. Show that $\dot{r} = \frac{dr}{dt} = r(1 - r)$ can be written as $(\frac{1}{r} + \frac{1}{1-r})dr = dt$ (*).
14. At time $t = 0$, the radius is equal to r_0 and at time $t = 2\pi$ the radius is equal to r_1. Integrate the equation (*) and find the discrete Poincare map $r_1 = P(r_0)$. Show that it is equal to

$$P(r_0) = \frac{1}{1 + (\frac{1}{r_0} - 1)e^{-2\pi}} \tag{4}$$

15. Sketch the the graph of the function $P(r_0)$. Use the graph to argue for the value of the fixed point $r^* = P(r^*)$, obtained as the limit $r^* = \lim_{n\to\infty} P^n(r_0)$.
16. Calculate the stability of the map in the fixed point r^*.