Exam for Dynamical Systems and Chaos, 16 April 2010.
Duration: 3 hours. Books, notes and computers are allowed.
Exam consists of 3 exercises. The questions are equally weighted except question 6 that counts half.
Answers can be written in Danish and English. Pencil is allowed
Results and solutions are posted on the home page.

I.

For the differential equation

$$\dot{x} = f(x) = r\ \cos x\ -\ \sin\ 2x = \cos x(r - 2\sin x) \tag{1}$$

1. $\dot{x} = 0 \Rightarrow x^* = \pi/2$ or $x^* = -\pi/2$ are always fixed points for any value of r.
2. $f'(x^*) = -r\sin x^* - 2\cos\ 2x^*$. $f'(\pi/2) = -r + 2 = 0 \Rightarrow r_{c1} = 2$. $f'(-\pi/2) = r + 2 \Rightarrow r_{c2} = -2$.
3. $\dot{x} = 0 \Rightarrow r\ \cos x^* - \sin\ 2x^* = \ = \cos x^*(r - 2\sin\ x^*) = 0 \Rightarrow \sin\ x^* = \frac{r}{2}$.
4. $f'(x^*) = -r\sin x^* - 2\cos\ 2x^* = -r\sin x^* - 2(1 - 2\sin^2 x^*) = \frac{r^2}{2} - 2$. The new fixed points exist only in the interval $r \in [-2; 2]$ and since $f'(x^*) \leq 0$ in this interval they are always stable.
5. $r \simeq 2 \Rightarrow x^* \simeq \pi/2 + x \Rightarrow \sin(\pi/2 + x) \simeq 1 - \frac{x^2}{2} = \frac{r}{2} \Rightarrow x = \pm\sqrt{2 - r} \Rightarrow x^* \simeq \pi/2 \pm \sqrt{2 - r}$. $(r \leq 2)$.
6. Pichfork bifurcation, super-critical.

   Fixed point $x^* = \pi/2$ stable for $r \geq 2$ (full line) and unstable for $r < 2$ (dashed line). Fixed point $x^* = -\pi/2$ stable for $r \leq -2$ (full line) and unstable for $r > -2$ (dashed line). Fixed points $\sin\ x^* = \frac{r}{2}$ in $r \in [-2; 2]$ are always stable (full). Around r_{c1}, r_{c2} bifurcated fixed points vary as squareroots. One branch goes through $(0, 0)$. The other 'jumps' from $(0, \pi)$ to $(0, -\pi)$.

II.

7. $\dot{x} = 0$ and $\dot{y} = 0 \Rightarrow (x^*, y^*) = (0, 0), (-2, \sqrt{r}), (-2, -\sqrt{r})$ are fixed points.
8. There is a bifurcation point at $r_c = 0$.
9. For $(x^*, y^*) = (0, 0)$ the eigenvalues are $\lambda = -r, -2$. It is a stable node for $r > 0$ and a saddle point for $r < 0$.

   For $(x^*, y^*) = (-2, \pm\sqrt{r})$ the eigenvalues are $\lambda = \pm 2\sqrt{r}$. For $r > 0$ (when the fixed points exist) both fixed point are saddle points.
10. It is a stable node for $r > 0$ and a saddle point for $r < 0$.

III.

11. $x_n^* = 0, \frac{1}{2}, 1$. All derivatives are ± 3 so they are all unstable.
12. Under the mapping: $x \to 3x \to 9x - 2 = x \Rightarrow \frac{1}{4} \to \frac{3}{4} \to \frac{1}{4}$
13. Stability: $f'(\frac{1}{5}) \cdot f'(\frac{3}{5}) = -9$, unstable
14. Under the mapping: $x \to 3x \to 9x \to 27x \to 2 - 81x = x \Rightarrow \frac{1}{41} \to \frac{3}{41} \to \frac{9}{41} \to \frac{27}{41} \to \frac{1}{41}$.