Svar: Eksamensopgaver til Fysik 222, 30. januar 2002.

I.

1. $\dot{x} = 0 \Rightarrow y = 0$. $\dot{y} = 0$ and $y = 0 \Rightarrow x = 0$.
   $\lambda = (\mu \pm \sqrt{\mu^2 - 4})/2$. Real part: $\mu/2$ vanishes at $\mu = 0$.
2. Eigenvalues are complex: oscillating solutions.
3. $x = r\ cos\theta, y = r\ sin\theta, r^2 = x^2 + y^2$.
   $\dot{r} = h_1(r, \theta) = r(\mu - r)sin^2\theta$
   $\dot{\theta} = h_2(r, \theta) = -1 + (\mu - r)\ cos\theta\ sin\theta$.
4. $\mu \ll 1$ and $r \ll 1 \Rightarrow \dot{\theta} = -1 \Rightarrow \theta(t) = -t + \theta_0$
5. As $\theta(t) = -t + \theta_0 \Rightarrow < sin^2\theta >= 1/2$ (see Strogatz p. 224). This leads to: $\dot{r} = h(r) = r\frac{1}{2}(\mu - r)$.
   $\dot{r} = 0 \Rightarrow r^* = 0$ or $r^* = \mu$
6. $h'(r) = \frac{1}{2}\mu - r$. $h'(0) = \frac{1}{2}\mu$, $h'(\mu) = -\frac{1}{2}\mu$.
7. $r^* = 0$ stable for $\mu < 0$ and unstable for $\mu > 0$. $r^* = \mu$ unstable for $\mu < 0$ and stable for $\mu > 0$.
8. The bifurcation is NOT a *generic* Hopf bifurcation because the radius of the limit cycle grows proportional to μ and not to $\mu^{1/2}$ (see Strogatz p. 251).

II.

9. $f_r(x_n) = x_n \Rightarrow x_n^* = 0,\ \sqrt{1 - r},\ \ -\sqrt{1 - r}$.
10. $f_r'(x_n) = r + 3x_n^2$. $f_r'(0) = r$. $x_n^* = 0$ stable in $[-1, 1]$, i.e. $r_- = -1, r_+ = 1$.
11. $f_r'(\pm\sqrt{1 - r}) = 3 - 2r$. Fixed points $\pm\sqrt{1 - r}$ are always unstable (when $r \leq 1$ where they exist).
12. At $r = r_+ = 1$ a pitchfork bifurcation takes place where an unstable fixed point bifurcates into two unstable fixed points and becomes stable itself.
13. At $r = r_- = -1$ a period doubling takes place for $x_n^* = 0$ (the eigenvalue is -1).
14. $f_r^2(x^c) \simeq (1+2\epsilon)x^c - (2+4\epsilon)(x^c)^3 = x^c$. It is a second order equation with solutions $x^c \simeq \pm\sqrt{\epsilon}$.