Exam for Dynamical Systems and Chaos, 16 April 2010.
Duration: 3 hours. Books, notes and computers are allowed.
Exam consists of 3 exercises. The questions are equally weighted except question 6 that counts half.
Answers can be written in Danish and English. Pencil is allowed
Results and solutions are posted on the home page.

I.

Consider the differential equation

$$\dot{x} = r \cos x - \sin 2x, \qquad -\pi < x \leq \pi \tag{1}$$

1. Show that $x = \pm\pi/2$ are fixed points for any value of the parameter r.
2. Show that both fixed points undergo a bifurcation as r is varied. Find the bifurcation points $r_{c1} > 0$ and $r_{c2} < 0$ of the fixed points.
3. Show that the bifurcations give rise to fixed points that satisfy

$$\sin x^* = \frac{r}{2} \tag{2}$$

4. Determine the stability of these fixed points using relation (2).
5. Close to the bifurcation point r_{c1} (i.e $x^* \simeq \pi/2$), Taylor expand relation (2) to second order and derive an approximation for the fixed points as a function of r below the bifurcation point ($r \leq r_{c1}$).
6. What type of bifurcation takes place at r_{c1}. Is it super- or sub-critical ? A similar bifurcation takes place at $r = r_{c2}$. Sketch the bifurcation diagram for all values of r and $x^* \in]-\pi;\pi]$ (full line: stable fixed points, dashed line: unstable fixed points).

II.

Consider the differential equation

$$\begin{aligned} \dot{x} &= xy^2 - rx \\ \dot{y} &= -xy - 2y \end{aligned} \tag{3}$$

where r is a parameter.

7. Find the fixed points for (3) in the (x, y) plane as a function of r.
8. Identify a bifurcation point and find the critical value of the parameter, r_c where the bifurcation takes place.
9. Determine the linear stability of the fixed points.
10. What happens for the fixed point $(0, 0)$ at r_c (describe in words). Sketch the flow around $(0, 0)$ for $r < 0$ and $r > 0$ respectively (use the eigen-values and -vectors).

III.

Consider the discrete mapping

$$x_{n+1} = f(x_n) = \begin{cases} 3x_n & \text{if} \quad x_n \leq 1/3 \\ 2 - 3x_n & \text{if} \quad 1/3 \leq x_n \leq 2/3 \\ 3x_n - 2 & \text{if} \quad 2/3 \leq x_n \end{cases} \tag{4}$$

11. Determine the fixed points of (4) and their stability.

    A cycle-point x_n^* can be associated with the letter L if $x_n^* \in]-\infty; 1/3]$, with the letter M if $x_n^* \in [1/3; 2/3]$ and with the letter R if $x_n^* \in [1/3; \infty[$.
12. Find the orbit points the two-cycle, where $x_n^* = f^2(x_n^*)$, of the mapping (4) (and different from the fixed point) which in terms of these symbols can be described as $LRLRLR$.....
13. Determine the stability of the two-cycle.
14. The mapping (4) has a four-cycle, where $x_n^* = f^4(x_n^*)$ (and different from the fixed point) which in terms of the symbols can be described as $LLLMLLLM$..... Determine the orbit points of this four-cycle.