Eksamensopgaver til Fysik 222, 20. juni 2005.
Varighed 3 timer. Hjælpemidler tilladt. Eksamen består af 3 opgaver.
Spørgsmål 1-16 indgår med lige vægt i bedømmelsen.
Eksamensresultat og løsninger opslås på hjemmesiden.

I.

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= \mu - xy^2 \\ \dot{y} &= xy^2 - y \qquad\qquad \mu \neq 0 \end{aligned} \tag{1}$$

1. Find the fixed point(s) for the system (1).
2. Determine the eigenvalues, $\lambda(\mu)$, as a function of μ for the fixed point.
3. Find the value of μ^2 where the fixed point changes from being a stable spiral to a stable node.
4. Describe how the behavior of the fixed point changes at the μ-value $\mu^2 = 3 - \sqrt{8}$.
5. Find the values of μ where there is a Hopf bifurcation.

II.

Consider the differential equation

$$\dot{x} = f(x, r) = r \ \tan x \ - \ \sin 2x, \qquad -\pi < x \leq \pi \tag{2}$$

6. Show that $x = 0$ and $x = \pi$ are fixed points for any value of the parameter r.
7. Show that both fixed points undergo a bifurcation as r is varied. Find the bifurcation point r_c.
8. Show that the bifurcations give rise to fixed points that satisfy

$$\cos^2 x^* \ = \ \frac{r}{2} \tag{3}$$

9. Determine the stability of these fixed points using relation (3).
10. For $x^* \simeq 0$ and close to the bifurcation point r_c, Taylor expand $\cos x$ to lowest (i.e. second) order and use relation (3) to derive an approximation for the fixed points as a function of r below the bifurcation point ($r \leq r_c$).
11. What type of bifurcations takes place at r_c. Are they super- or sub-critical ?
12. Sketch the bifurcation diagram for all values of r and $x^* \in]-\pi; \pi]$ (full line: stable fixed points, dashed line: unstable fixed points).

III.

Consider the discrete mapping

$$x_{n+1} = f(x_n) = x_n + r - x_n^2 \qquad\qquad r \geq 0 \tag{4}$$

13. Determine the fixed points for (4) and their stability.
14. Determine the x-value, x_s, of the super-stable fixed point.
15. Find the r-value where a period dobling takes place.
16. Write a second-order equation in r which determines the r-value of the super-stable two-cycle (Hint: x_s is part of a super-stable two-cycle).